Branched classical and quantum flow in two-dimensional Wannier-Stark systems

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The quantum and classical dynamics in a two-dimensional (2D) periodic potential influenced by a constant force have been of central interest since the early days of quantum mechanics. The first work in this area was by Wannier [1–3]. Later, Wannier [4] and Zak [5] discussed the energy spectrum of these systems and initiated a long debate about its nature. Wannier suggested that the spectrum should consist of an infinite number of equally spaced energy levels, the so-called Wannier-Stark ladders, whereas Zak proved that the spectrum is continuous. This paradox was solved when it was shown that both were right: The spectrum of a WS system is continuous and the energy ladders proposed by Wannier are embedded as ladders of resonant states.

Recently, a method introduced to analyze one-dimensional Wannier-Stark systems [6–8] was successfully extended and applied to two-dimensional (2D) systems [9]. In this paper we use this method to discuss the dynamics in a simple two-dimensional model describing, e.g., the potential created by two standing laser waves crossing at right angles. A discussion of the classical diffusion in such systems—but without the Stark field—can be found in [10]. For recent findings on the dynamics of electron gases in a 2D system see [11].

The structure of the paper is as follows. After a short introduction to the model in Sec. II, we discuss the classical dynamics and illuminate the transport phenomena which can be observed (Sec. III). The quantum dynamics is discussed in Sec. IV and compared to the classical results obtained before. We conclude with Sec. V.

II. THE 2D WANNIER-STARK SYSTEM

In the following we are going to consider the two-dimensional Wannier-Stark Hamiltonian

\[ \hat{H} = \frac{\hat{p}^2}{2} + V(\hat{r}) + \hat{F} \cdot \hat{r}, \quad \hat{r} = (\hat{x}, \hat{y}), \]  

(1)

where

\[ V(\hat{r}) = \cos \hat{x} + \cos \hat{y} - \epsilon \cos \hat{x} \cos \hat{y} \]  

(2)

is a doubly periodic potential with period \( 2\pi \) in each direction. The choice of equal periods in the two directions simplifies the following discussion. The two values \( \epsilon = 0 \) and \( \epsilon = -1 \) correspond to a separable potential (\( \epsilon = 0 \)) in which the classical dynamics is regular and a strongly coupled case (\( \epsilon = -1 \)) in which the classical dynamics shows chaotic behavior. The first choice realizes an “egg crate” potential, the second one a simple “quantum dot” potential. The choice \( \epsilon = -1 \) describes also an optical potential created by two lasers crossing at right angles or—in the classical case—the diffusion behavior for atoms or molecules on crystal surfaces [12]. The resulting potential is shown in Fig. 1.

For reasons which will be made clear when we reach the quantum mechanical part of this work, we have to make special assumptions about the direction of the applied field. It is given by

\[ \hat{F} = \frac{F}{\sqrt{q^2 + r^2}} (q \hat{x}, r \hat{y}), \]  

(3)

\[ q, r \in \mathbb{N}, \quad \gcd(q, r) = 1, \]  

(4)

i.e., \( q \) and \( r \) are coprime integers. We call this a rational direction of the field.

FIG. 1. Potential energy (2) for \( \epsilon = -1 \).
be more specific, we choose trajectories which begin their journey with equal momentum. To emphasize this, let us discuss an ensemble of phase space partitions. As expected, the system behaves strongly chaotic. To assign to the motion in the $x$-space.

The similar behavior of trajectories with almost identical initial conditions does not hold for the coupled case $\epsilon \neq 0$. To be more specific, we choose $\epsilon = -1$ in the following discussion. As expected, the system behaves strongly chaotic. To emphasize this, let us discuss an ensemble of phase space trajectories which begin their journey with equal momentum of absolute value $p_0 = 4$ aligned against the field direction (characterized by $q = 2$, $r = 1$, and $F = 0.16 \sqrt{2}$ in this example) and with starting points approximately in the same area (by holding the energy of the different particles fixed and therefore varying the absolute value of the initial momentum one arrives at similar results). All of these trajectories are not trapped up to a set of measure zero. Figure 2 shows the momentum distribution at time $t \approx 106$ of such an ensemble after these trajectories were scattered by the periodic potential—the distribution in coordinate space looks similar (Fig. 3)—at least with respect to the possible outgoing directions.

The distribution is clearly structured and the number of possible escape directions is limited. In fact, all trajectories travel either in the direction of the field or to the angles $\theta \in \{3\pi/4, \pi, 5\pi/4, 3\pi/2\}$. We will call these well separated directions in which the trajectories end up “channels.” A fundamental difference between the channel corresponding to the field direction (the “field channel”) and the other channels (the “symmetry channels”) exists. The boundaries of the possible momenta are sharp in the symmetry channels in comparison to the diffuse boundary of the distribution in the field direction. In configuration space the symmetry channels form pencil-shaped structures whereas the trajectories which travel in the field direction form a diffusive bulb. Thus we have to distinguish two types of free motion.

Let us discuss the origin of the formation of the different channels. We start with the trajectories moving along the negative $x$ direction. Such a particle is accelerated in the $x$ direction, but the motion in the $y$ direction is confined despite the fact that the confining potential wall goes to zero at regular intervals. Pictorially, the motion can be described as the oscillatory motion down a steep alley bounded by equidistant trees, the potential hills. To understand this confinement we consider the equation of motion for $y$:

$$\ddot{y} = \sin y (1 + \cos x) - F_y. \quad (6)$$

Because $x$ varies fast in comparison to $y$ [$x(t) = \omega(t)t + x_0$ with $\omega(t) = ct + \omega_0$], we write Eq. (6) in the form

$$\ddot{y} = \sin y - F_y + f(y,t) = - \frac{\partial V_0}{\partial y} + f(y,t), \quad (7)$$

where $f(y,t)$ takes into account the rapidly varying term $f(y,t) = \sin y \cos(\omega(t)t + x_0)$. The motion controlled by $f(y,t)$ can be separated from the overall behavior by writing $y$ in...
the form \( y = Y + \eta \), where \( \eta \) is assumed to be small and satisfies the equation of motion [13]

\[
\ddot{\eta} = f(y, t).
\] (8)

In the short time interval \([t_0, t_0 + T]\) defined by \( x(t_0 + T) = x(t_0) + 2\pi \lim_{t_0 \to \infty} T = 0 \), the function \( f(y, t) \) can be approximated by

\[
f(y, t) \approx \sin Y \cos(\tilde{\omega}t + x_0),
\] (9)

with \( \tilde{\omega} = \omega(t_0) + \omega_0 \). Therefore an approximate local solution to Eq. (8) is

\[
\eta = -\frac{1}{\omega^2} f(Y, t).
\] (10)

By expanding Eq. (7) around \( y = Y \) and time averaging it follows that \( Y \) satisfies the equation

\[
\dot{Y} = -\frac{\partial V_0}{\partial Y} + \frac{\partial f}{\partial Y} = -\frac{\partial V_0}{\partial Y} - \frac{1}{\omega^2} \frac{\partial f}{\partial Y}.
\] (11)

Therefore the \( Y \) motion is described by the effective potential

\[
V_{\text{eff}} = V_0 + \frac{1}{\omega^2} \frac{1}{2} \sin^2 Y,
\] (12)

which supports bounded trajectories for relatively small amplitudes. Because \( \eta \) is small, this potential describes also the overall behavior of \( y \). Thus we can expect to find trajectories which are periodic in the \( y \) coordinate with a bounded amplitude.

Let us briefly discuss the validity of the substitution \( \omega(t) \rightarrow \tilde{\omega} \) in Eq. (9). We can reintroduce the time-dependent \( \omega \) in Eq. (12) in a simple way by writing

\[
V_{\text{eff}} = V_0 + \frac{1}{2 \omega^2(t)} \sin^2 Y.
\] (13)

This shows that in the long time limit where \( \omega(t) \) approaches infinity the last term can be neglected. Finally, one can argue that the time-dependence of \( \omega \) should prevent the time average in Eq. (12) where we replaced \( \cos^2 \omega t \) by \( 1/2 \). But this formula is approximately correct even for the case \( \omega = \omega(t) = c t + \omega_0 \) as can be seen from

\[
\cos^2[\omega(t)t] = \frac{1}{T(t)} \int_t^{t + T(t)} \cos^2[\omega(t')t'] \, dt'
\]

\[
= \frac{1}{2T(t)} \int_t^{t + T(t)} \{1 + \cos[2\omega(t')t']\} \, dt'
\]

\[
= \frac{1}{2} + \frac{1}{4T(t)} \int_t^{t + T(t)} (e^{2i\omega(t')t'} + e^{-2i\omega(t')t'})
\]

\[
\approx \frac{1}{2},
\] (14)

where \( T(t) \) denotes the local period. The last approximation can be justified by the method of stationary phase. The stationary point \( t_s \) is given by \( t_s = -\omega_0/2c \) for both exponential functions. Therefore the last integral contributes significantly only for \( t_s \in [t, t + T(t)] \) and it is expected to be small for all other time intervals. As an example, Fig. 4 shows the time dependence of the \( y \) coordinate in the full potential and the effective potential (12) in comparison.

Until now we have only explained the existence of the channel along the \( x \) direction due to dynamical trapping. For symmetry reasons the same argument holds for trajectories traveling along the \( y \) axis which show an oscillating behavior in the \( x \) coordinate. By rotating the coordinate system by the angle \( \pi/4 \) which leads to the potential

\[
V(x', y') = \frac{1}{2} \cos(2x') + \frac{1}{2} \cos(2y') + 2 \cos x' \cos y'
\] (15)

in the new coordinates \((x', y')\), we see that the channels at \( \theta = 3\pi/4 \) and \( \theta = 5\pi/4 \) can be explained in a similar way.

Other escape directions are blocked by the considerably large width of the potential. These channels can be opened by considering, e.g., the potential

\[
V_{m,n}(x, y) = 4 \cos^2(\frac{x}{2}) \cos^2(\frac{y}{2}) - 1,
\] (16)

which is equivalent to Eq. (2) for \( m = n = 1 \). For \( m \) or \( n \neq 1 \) the maxima of this potential, i.e., the location of the peaks, stay the same but the valleys get broader. Numerical calculations show that this opens new stable escape directions as expected.

Let us now return to the discussion of the potential (2). The chaotic behavior of the system can be visualized if we assign a different shade of gray to each decay channel and mark the initial coordinates according to the gray tone of
their final decay channel. Such a plot is shown in Fig. 5 for the trajectories whose final fate is depicted in Fig. 2. One observes a fractal-like distribution of the different shades. The areas with different shades of gray are strongly entwined and homogeneous regions are rare. A crude overall structure which results from the shape of the potential is present in the form of elliptical curves. To emphasize the fractal structure of Fig. 5, we show two subsequent magnifications in Figs. 6 and 7. In both images, we find a mixture of regular parts and areas in which the final conditions vary strongly. This sequence of magnifications can be continued ad infinitum.

It is illuminative to have a look at the time \( \tau \) in which different trajectories with almost identical initial conditions stay in the vicinity of the scattering region. The way in which this dwelling time is defined is somewhat arbitrary. Here we choose it as the length of the time interval in which the magnitude of the momentum is smaller than a given value. The distribution of the dwelling time \( P(\tau) \) should follow an exponential law if the scattering is chaotic:

\[
P(\tau) \sim e^{-\tau/\tau_c}.
\]

Figure 8 shows the appropriate distribution for the ensemble studied above. The semilogarithmic plot in the inset shows that the decay is indeed exponential with \( \tau_c \approx 7.3 \). This is another indication of chaotic scattering. In the separable case \( (\epsilon = 0) \) all examined trajectories exhibit approximately the same dynamics and their dwelling times cluster around the value \( \tau_c \approx 20 \).

IV. QUANTUM DYNAMICS AND QUANTUM-CLASSICAL CORRESPONDENCE

Before we discuss the 2D case, we will briefly recall the basic features of the 1D theory \([6–8,14]\) which is described by the Hamiltonian

\[
\hat{H} = \frac{p^2}{2} + V(\hat{x}) + F\hat{x}, \quad V(x) = V(x + 2\pi).
\]
The key operator which governs the dynamics is the time evolution operator $\hat{U}(T_B)$ over the so-called Bloch time $T_B = h/F$. It is easy to check that $\hat{U}(T_B)$ commutes with the translation operator over one period of the potential. The eigenstates of $\hat{U}(T_B)$, the so-called Wannier-Bloch or Floquet-Bloch states, are therefore simultaneously eigenstates of the space translation operator over $2 \pi$. Since the latter operator is unitary, the corresponding eigenvalue $\lambda$ can be written in the form $\lambda = \exp(i 2 \pi \kappa)$, where $\kappa \in [-1/2,1/2]$ is the so-called quasi momentum or Bloch index. The continuous time evolution of the eigenstates follow the equation

$$\psi_{\alpha,\kappa}(x,t) = e^{-iE_{\alpha,\kappa}t/h} \psi_{\alpha,\kappa-F/t\hbar},$$

where $\alpha \in \mathbb{Z}$ is a band index which sorts the different states according to their life time. The eigenvalues are given by

$$E_{\alpha,\kappa} = E_{\alpha} + 2 \pi F l - i \kappa \frac{T_B}{2}, \quad l \in \mathbb{Z},$$

forming different sets of ladders of resonances, the so-called Wannier-Stark ladders. Note that $E_{\alpha,\kappa}$ is independent of the Bloch index $\kappa$, which leads to flat bands.

Let us now proceed to the 2D case. Provided that the rationality condition (4) for $q$ and $r$ is satisfied, it is possible to define a 2D Bloch time which is given by the least common multiple of the two 1D Bloch times $T_x = h/F_x$ and $T_y = h/F_y$:

$$T_B = q T_x = r T_y = \sqrt{q^2 + r^2} h/F.$$  

Note that $T_B$ defines the characteristic time scale for the dynamics and depends strongly on the direction of the field $F$. Introducing the wave function $\tilde{\psi} = \exp(i F \cdot r t/h) \psi$, we arrive at the Schrödinger equation

$$i \hbar \frac{\partial \tilde{\psi}}{\partial t} = \hat{H} \tilde{\psi},$$

where $\hat{H}$ is given by

$$\hat{H}(t) = \frac{(\hat{p}_x - F_x t)^2}{2} + \frac{(\hat{p}_y - F_y t)^2}{2} + V(\hat{x}, \hat{y}).$$

The time evolution operator $\tilde{U}$ for this system is connected to the initial one by

$$\tilde{U}(t) = \exp \left( - \frac{i}{\hbar} (F \cdot r) t \right) \tilde{U}(t)$$

or, for $t = T_B$,

$$\tilde{U}(T_B) = e^{-i q \hat{x} \cdot \hat{r} \left( \frac{T_B}{h} \right)} \exp \left( - i \frac{r}{\hbar} \int_0^{T_B} dt \hat{H}(t) \right).$$

From an analysis of this time evolution operator, the following results were obtained [9]: The energy spectrum consists of different Wannier-Stark ladders as in the 1D case. The splitting between two neighboring energy levels in one ladder given by $\Delta E = 2 \pi F/(\sqrt{q^2 + r^2})$ is highly sensitive to changes in the direction of the field due to the denominator. The real part of the dispersion relation $E(\kappa)$, where $\kappa$ is a Bloch index, is constant in the field direction and periodic orthogonal to the field in quasi-momentum space. The resonance width $\Gamma$ also shows a highly nontrivial dependence on the field direction. Therefore a complicated behavior of the survival probability is expected when the direction of the field is varied.

Equation (25) is the vantage point for our numerical calculation. Numerically, we take a basis set of plane waves

$$\langle r|n \rangle = \frac{1}{2\pi} e^{i r \cdot n}, \quad n \in \mathbb{Z} \otimes \mathbb{Z}$$

for a representation of $\tilde{U}(T_B)$. The propagator obtained in this way is used to evolve a Gaussian wave packet $|k_0, r_0\rangle$.

$$\langle k|k_0, r_0\rangle = \frac{1}{\sqrt{\pi \gamma}} \exp \left( - \frac{1}{2\gamma} (k - k_0)^2 - i r_0 \cdot \left( \frac{k - k_0}{2} \right) \right).$$

with $\langle k_0, r_0|k_0, r_0\rangle = 1$ and $\langle k_0, r_0|\hat{p}|k_0, r_0\rangle = h k_0$.

As an example, let us discuss the case considered in Sec. III ($q=2, r=1$, and $F=0.16/\sqrt{2}$ with strong coupling $\epsilon = -1$) for $h=2$. Figure 9 shows the probability distribution of the wave packet with initial conditions $k_0=0, r_0=0$ and $\gamma = \pi$ at times $t=0$ and $t=5T_B$ in momentum space, i.e., we start a Gaussian quantum ensemble with initial zero average velocity located at a maximum of the potential. In this figure the direction of the force points approximately away from the reader. The center of the wave packet coincides with a maximum of the potential, and, because of $k=0$, the wave packet starts directly in the interaction region. The distribution develops a complicated structure. The main part of the wave packet travels in the direction of the applied field. Smaller parts move along the axes. This behavior will be analyzed in the next section in more detail.

FIG. 9. Evolution of an initially Gaussian wave packet at times $t=0$ (left) and $t=5T_B$ (right).
Because of the static field and quantum tunneling, the system discussed cannot support bound states. Every initial state will eventually decay exponentially into the third quadrant in coordinate space or, equivalently, momentum space. Figure 10 shows the probability $W$ of a particle described by the Gaussian wave packet to remain in the region with $k \leq 3$ as a function of time. As expected, the staying probability $W$ drops off exponentially $\frac{W(t)}{W(0)} = \exp(-\Gamma t/\hbar)$, where the decay rate is approximately $\Gamma \approx 0.108$. This should be compared to the widths of the Wannier Bloch states. By diagonalizing $\hat{U}(T_B)$ we find that the widths of the two most stable states are $\Gamma_0 \approx 0.094$ and $\Gamma_1 \approx 0.218$. We conclude that the initial wave packet populates primarily the ground state, i.e., the most stable resonance. Note that the asymptotic long-term behavior of the survival probability is always given by the decay rate $\Gamma_0$ of the most stable state, but here this holds for small times $t$, too.

Finally, we compare the quantal and classical behavior of the system. The main questions will concern the decay directions and the associated probabilities. For this purpose we considered the field directions $q \in \{2,3,4\}$ with $r = 1$, where $F = 0.16\sqrt{2}$ and $\hbar = 2$ are held fixed. The quantum mechanical results are obtained by propagating the wave packet (27) mentioned as an example in the previous section. For the classical case we consider the corresponding initial phase space distribution

$$\rho(p,r) = \frac{1}{\pi^2} \exp \left( - \frac{1}{\gamma} \frac{(p-p_0)^2}{\gamma} \right)$$  

with $\gamma = \gamma \hbar^2$. Both distributions are propagated for the same time interval $t$ and afterwards the distribution of the directions of the velocities are analyzed.

Figure 11 shows the appropriate histograms for $q = 2$ corresponding to the field direction $\theta_F = \arctan(1/2) + \pi = 1.15\pi$ at $t = 6T_B$. The bars indicate the decay probability in the direction $\theta$. The classical result (bottom) shows the well separated decay channels already familiar from Fig. 2. The main part of the wave packet decays into the direction of the field $\theta_F \approx 1.15\pi$ and at an angle of $\theta = 5\pi/4$, but the other directions mentioned in Sec. III are clearly present, too. This can be compared with the quantum mechanical result (top). Due to tunneling, the different decay channels are not so well separated—especially in the vicinity of the direction of the applied field—and the decay channel at $\theta = 3\pi/4$ is missing due to destructive interference. Nonetheless we find a correspondence between the locations of the peaks in the quantum and classical case. The heights of the quantum and classical peaks are approximately the same along the field direction and at the angle $\theta = \pi$ whereas the direction $\theta = 3\pi/2$ is present in the quantum case but strongly suppressed in comparison to the classical one.

For $q = 3$ (Fig. 12) which corresponds to $\theta_F \approx 1.1\pi$, the
The difference between the angles of the two prominent decay directions is larger which results in an improved separation of the respective peaks in the histogram. The main part travels again in the field direction, but the side peaks are present again, with the exception of the direction $u = 3\pi/4$ in the quantum case. This is also true for the final example $q = 4$, i.e., $\theta = 1.08\pi$ (Fig. 13). In this case the quantum distribution is washed out along the direction of the field but parts of the wave packet are again transported along the axes. The peak which corresponds to $\theta = 5\pi/4$ is embedded in the flanks of the main peak and hardly distinguishable from this background.

To check the validity of the results described we varied the initial location of the propagated wave packet. The formation of channels is independent of the starting point, all directions mentioned before are still present when we place the center of the distribution not on the top of a hill as before, but, pictorially speaking, on the hillside or at the bottom of a valley. The only change observed is a variation of the intensity distribution between the different channels as expected.

V. CONCLUSION

We studied the quantum dynamics of Gaussian wave packets in 2D Wannier Stark systems and compared the results obtained with the dynamics of corresponding classical ensembles. In both cases a formation of well separated decay channels can be observed which resembles the coherent branched flow in a 2D electron gas [11]. Due to the higher symmetry in our model, the formation of the different branches is even more prominent. In the classical case the stability and formation of the observed structures were explained by dynamical trapping. The intensity distribution between the different decay channels is similar in the quantum and the classical case. It should be possible to see these structures in appropriate experiments involving, e.g., cold atoms in a 2D optical lattice.