DERIVATION OF QUASIDETERMINISTIC FOKKER–PLANCK DYNAMICS

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For a diffusion process with constant diffusion coefficient we give a derivation of an approximation of the transition probability density by means of the induced measure and the concept of the most probable path. In close analogy to semiclassical quantum mechanics the transition probability density is completely expressed in terms of the action of the most probable path.

The physical significance of the functional integral representation of the transition probability density of a diffusion process is based on the fact that this representation allows insight into the connection between the behavior of the single realizations of the process and quantities which are determined by certain sets of realizations.

Recent derivations of the path integral have been set up by means of the discretisation of the stochastic differential equation [1–3], by means of an iteration of the short time propagator of the Fokker–Planck equation [4–6] or equivalent operator ordering techniques [7–9] and by means of the induced measure [10–12].

Due to the introduction of a description-dependent integration prescription some of the methods mentioned do not allow to carry over information from the sample paths to the integrand. As a result, a free parameter may occur in the integrand [3,5,9]. The advantage of the procedure by means of the induced measure is clearly demonstrated by the fact that this method permits an unambiguous determination of the Onsager–Machlup function as well as a physically suggestive definition of the most probable path of a diffusion process.

Whereas the determination of the Onsager–Machlup function and the differential equation for the most probable path has been set up in [11], the purpose of this note is to show how a simple approximation of the transition probability density by means of the induced measure and the concept of the most probable path can be obtained.

Let us consider a one-dimensional diffusion process $X_s, s \in [0, t]$ with constant diffusion $c$

$$dX_s = f(X_s) \, ds + c \, dW_s, \quad X_0 = x_i.$$  \hspace{1cm} (1)

The measure $\mu_X$ induced by $X_s$ is defined on the space of continuous functions $C(x_i)$

$$C(x_i) = \{ \xi | \xi \in C([0, t]), \xi(0) = x_i \}.$$  \hspace{1cm} (2)

Summing up all paths $\xi(s)$ connecting $x_i = \xi(0)$ and $x_f = \xi(t)$ we obtain the transition probability density (t.p.d) $p(x_i|t, x_f)$ of $X_s$

$$p(x_i|t, x_f) = \int_{C(x_i, x_f)} \mu_X[\xi],$$  \hspace{1cm} (3)

where

$$C(x_i, x_f) = \{ \xi | \xi \in C(x_i), \xi(t) = x_f \}.$$  \hspace{1cm} (4)

As $\mu_X$ is absolutely continuous with respect to the measure $\mu_{\tilde{W}}$ induced by the modified Wiener process $\tilde{W}_s, s \in [0, t]$, $d\tilde{W}_s = c \, dW_s$  \hspace{1cm} (5)

the t.p.d. can be expressed in terms of a Wiener integral

$$p(x_i|t, x_f) = \int_{C(x_i, x_f)} \mu_X[\xi] \frac{d\mu_X[\xi]}{d\mu_{\tilde{W}}}.$$  \hspace{1cm} (6)
Here $\frac{d\mu_X}{d\mu_{\hat{W}}}[\xi]$ is the Radon–Nikodym derivative of $\mu_X$ with respect to $\mu_{\hat{W}}$ [11]

$$\frac{d\mu_X}{d\mu_{\hat{W}}}[\xi] = \exp \left\{ U(\xi(t)) - U(\xi(0)) \right\}$$

$$- \frac{1}{2} \int_0^t ds \left\{ c^{-2} f^2(\xi) + \frac{df}{dx}(\xi) \right\},$$

where

$$U(y_2) - U(y_1) = c^{-2} \int y_2^{y_1} dz f(z).$$

Let $x(s), x \in C^2([0, t])$ be the most probable path of the diffusion process $X_s$. As has been shown in [11] the Onsager–Machlup function

$$OM[\dot{x}, x] = \frac{1}{2} \left\{ c^{-2} [\dot{x} - f(x)]^2 + \frac{df}{dx}(x) \right\}$$

acts as a lagrangian for this path.

By means of the translation $T$

$$T : C(x_i) \rightarrow C(x_j), \quad T\dot{x} = \dot{x} - x + x_i$$

the area of integration is transformed into $C(x_j, x_i)$ and we obtain

$$p(x_i|t, x_j) = \int \frac{d\mu_{\hat{W}}}{\mu_{\hat{W}}}[\xi]$$

$$\times \frac{d\mu_X}{d\mu_{\hat{W}}}[\xi + x - x_j] J_{\hat{W}}[\xi, -x + x_j].$$

$J_{\hat{W}}$ denotes the Radon–Nikodym derivative of $\mu_{\hat{W}} - 1$ with respect to $\mu_{\hat{W}}$ and is given by [11]

$$J_{\hat{W}}[\xi, -x + x_j] = \exp \left\{ -c^{-2} [\xi(t)\dot{x}(t) - \xi(0)\dot{x}(0)] \right\}$$

$$- \frac{1}{2} \int_0^t ds \left\{ -2 \xi\ddot{x} + \dot{x}^2 \right\}.$$  

(12)

Explicitly the functional integral representation of the t.p.d. reads

$$p(x_i|t, x_j) = \int \frac{d\mu_{\hat{W}}}{\mu_{\hat{W}}}[\xi] \exp \left\{ -L_X[\xi] \right\}$$

where

$$L_X[\xi] = c^{-2} \left\{ \xi(t)\dot{x}(t) - \xi(0)\dot{x}(0) \right\}$$

$$- U(x_j) + U(x_i) + \frac{1}{2 c^2} \int_0^t ds \left\{ \dot{x}^2 - 2 \dot{x}\ddot{x} \right\}$$

(14)

$$+ f^2(\xi + x - x_j) + c^2 \frac{df}{dx}(\xi + x - x_j)$$

This functional can be expanded in terms of the deviations from the most probable path $Tx \equiv x_i$

$$L_X[\xi] = \int_0^t ds OM[\dot{x}, x]$$

$$+ \int_0^t ds(\dot{x} - x_j) \left\{ - \frac{d}{dt} \frac{\partial OM}{\partial \dot{x}} + \frac{\partial OM}{\partial x} \right\} [\dot{x}, x]$$

(15)

$$+ \sum_{n=2}^\infty \int_0^t ds \frac{1}{n^2} (\xi - x_j)^n \frac{\partial^n OM}{\partial x^n} [x].$$

The first term gives the action of the most probable path

$$\Phi(x_i, x_j, t) = \int_0^t ds OM[\dot{x}, x]$$

(16)

and the second term vanishes due to the property that the integrand contains the Euler–Lagrange equations for the most probable path

$$\ddot{x} = f(x) \frac{df}{dx}(x) + c^2 \frac{d^2 f}{dx^2}(x).$$

(17)

For an approximation of the t.p.d. we neglect terms higher than of third order in the expansion (15) and obtain

$$p^{QD}(x_i|t, x_j) = w(0|t, 0) \exp \left\{ -\Phi(x_i, x_j, t) \right\}.$$  

Here $p^{QD}(x_i|t, x_j)$ denotes the quasideterministic approximation and

$$w(0|t, 0) = \int \frac{d\mu_{\hat{W}}}{\mu_{\hat{W}}}[\xi] \exp \left\{ -\int_0^t ds \frac{1}{2} \xi^2 \frac{\partial^2 OM}{\partial x^2} [x] \right\}$$

(19)

defines the transition probability density of a diffusion process referring to the Bloch equation of a harmonic oscillator with time-dependent frequency [13].
The gaussian functional integral (19) can be evaluated by methods developed in [14]

\[ w(0|t, 0) = \left\{ 2\pi e^{2 \hat{x}(0) \hat{x}(t)} \int_0^t ds \hat{x}^{-2} \right\}^{-1/2} \]  

(20)

Using the results of [15], (20) can be easily expressed in terms of the action

\[ w(0|t, 0) = \left\{ -\frac{1}{2\pi} \frac{\delta^2 \Phi(x_f, x_f, t)}{\delta x_i \delta x_f} \right\}^{1/2} \]  

(21)

Finally, combining (18) and (21) we obtain the approximation of the t.p.d.

\[ p_{QD}(x_i|t, x_f) = \left\{ -\frac{1}{2\pi} \frac{\delta^2 \Phi}{\delta x_i \delta x_f} \right\}^{1/2} \times \exp \{-\Phi(x_i, x_f, t)\} \]  

(22)

The same result has been obtained in [15,16] by completely different methods. The advantage of the procedure developed here is based on the fact that a description-dependent representation is avoided from the very beginning. By means of the induced measures and the concept of the most probable path a straightforward derivation of the approximation is possible.

The approximate t.p.d. \( p_{QD} \) is solely determined by the underlying family of the deterministic most probable paths, which reveals the quasideterministic nature of Fokker–Planck dynamics in the limit considered here.

Evidently the approximation is good if the integrand is sharply peaked around the most probable path. This is fulfilled if \( \delta(0) \partial M/\partial x^n \) is negligibly small for \( n \geq 3 \). On the other hand this can be achieved if the diffusion \( c \) is very small and if

\[ \frac{1}{2c^2} \int_0^t ds (\hat{x} - f(x))^2 \gg \frac{1}{2} \int_0^t ds \frac{df}{dx} (\hat{x} + x - x_i) \]  

(23)

is fulfilled.

Contrary to the arguments given in [15] the approximation of the t.p.d. in the case of a small diffusion parameter only makes sense if inequality (23) holds. In this situation the term containing the derivative of the drift in (14) can be neglected. Physically this corresponds to a system which is weakly disturbed by irregular forces and the most probable path coincides with the phenomenological equation \( \dot{x} = f(x) \), which has been pointed out in [17]. It should be noted, that for the Ornstein–Uhlenbeck process (linear drift) \( p_{QD} \) of eq. (22) is identical to the exact result for the t.p.d.

Within the range of the approximation the Chapman–Kolmogoroff equation is valid for \( p_{QD} \). With regards to the fact that the integrand is assumed to be sharply peaked it is obvious that the most probable path from \( (x_i, 0) \) to \( (x_f, t) \) is the continuation of the most probable path from \( (x_i, 0) \) to \( (x_m, t_m) \) where \( 0 < t_m < t \) and that the most probable path from \( (x_m, t_m) \) to \( (x_f, t) \) is the restriction of the most probable path from \( (x_i, 0) \) to \( (x_f, t) \) to the interval \( [t_m, t] \). As the integrands in the functional integral representations for \( p_{QD}(x_i|t, x_f) \) and \( p_{QD}(x_f|t - t_m, x_f) \) constitute the integrand for \( p_{QD}(x_i|t - t_m, x_f) \) an integration with respect to \( x_m \) yields the Chapman–Kolmogoroff equation.

The structure of the approximate expression (22) is identical to the well-known semiclassical approximation to the quantum-mechanical propagator \( K(x_f, t; x_i, 0) \) (see for instance refs. [18,19]), apart from the fact that the exponent in (22) is real and not imaginary. This close analogy suggests that the methods and results of semiclassical mechanics will also apply to the analysis of Fokker–Planck dynamics.

Concerning the dynamics which is defined by the class of diffusion processes with constant diffusion coefficient we distinguish three levels of description:

(i) Fokker–Planck dynamics which describes the behavior of the system on a detailed statistical level.

(ii) Quasideterministic dynamics which is defined by the approximation of the t.p.d. given here. This level of description closely resembles the semiclassical limit of quantum theory.

(iii) Deterministic dynamics which is described by the equations of motion for the most probable paths.

References
