The kicked rotor: computer-based studies of chaotic dynamics

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The kicked rotor is a prototype of a classical nonlinear system with mixed regular and chaotic behavior. The dynamics reduces to a simple two dimensional area preserving mapping in phase space, which can be easily understood. A computer program is discussed in some detail which supports numerical experiments with this kicked rotor and related systems. Such a kicked rotor is, however, not only an academic toy system but also a basis for recent experimental research in quantum dynamics.

I. INTRODUCTION

Computers are an ideal tool to explore nonlinear systems and to demonstrate the intricate and often unexpected features of chaotic dynamics. Such systems are typical, at least in the world of classical mechanics, but nevertheless most textbooks mainly discuss the (rare) examples of regular motion because they allow an analytical analysis. For chaotic systems, the computer serves as a fast and efficient tool to generate a numerical solution of the equations of motion and therefore allows to explore their behavior and to investigate hidden details which are hard or even impossible to analyze by pure analytical methods. In particular a visualization of the results of a numerical solution is important and almost indispensable. Popular examples are the beautiful graphical images of fractal structures, as for instance found in the Mandelbrot set. Such computational experiments can, of course, not replace a rigorous mathematical analysis but they definitely provide valuable information.

As an example, the kicked rotor dynamics, which is explored in some detail in the following, illustrates this approach. This system provides a model with a simple mathematical structure. It can be used in mathematics to introduce graphical methods for studying mappings and recursion equations. In physics, it helps to introduce the concept of a phase space and to demonstrate its usefulness. Many general aspects of chaotic dynamics can be investigated. After all, the script language used in the program may be of interest as a very first introduction to computing, even for those having difficulties to enjoy the beauty of physics.

It should be pointed out, that the kicked rotor serves not only as an academic toy system for demonstrating chaotic dynamics. It provides also a basis for recent experimental research in quantum dynamics investigating, e.g., the non trivial relation between the nonlinear (chaotic) dynamics in the classical world and the quantum regime, which is governed by linear theories. The corresponding kicked quantum rotor has many physical realizations, including particle beams in an accelerator, atoms or molecules excited by microwaves [1, 2], as well as ultracold atoms subjected to a pulsed standing wave of near resonant light [3–5].

II. KICKED SYSTEMS AND DISCRETE MAPPINGS

Many systems of interest in the study of chaotic dynamics can be described by discrete mappings, which can be easily explored numerically. Here we will confine ourselves to the two-dimensional case. We use the variables \( q \) and \( p \), which may (but need not) appear as canonical coordinate and momentum. The mapping

\[
\vec{p} = f(p, q), \quad \vec{q} = g(p, q)
\]

maps the \((p, q)\)-plane, the ‘phase space’, onto itself. For a general discussion, we will denote the mapping by \( \mathbf{M} \):

\[
(p, q) \xrightarrow{\mathbf{M}} (\vec{p}, \vec{q}).
\]

In many applications, for example in Hamiltonian systems, the mapping \( \mathbf{M} \) is area preserving, i.e., the Jacobian determinant \( J \) which relates the phase space areas, \( \Delta \vec{p} \Delta \vec{q} = J \Delta p \Delta q \), is equal to one:

\[
J = \left| \frac{\partial (\vec{p}, \vec{q})}{\partial (p, q)} \right| = 1.
\]

Starting at a point \((p_0, q_0)\) and iterating the mapping (1) generates a sequence of points, \((p_n, q_n), n = 0, 1, 2, \ldots\), the ‘trajectory’ or ‘orbit’.

Such discrete mappings arise in various ways:

- (a) as a purely mathematical object, as for instance a linear mapping of the plane described by a matrix equation, or the celebrated Mandelbrot mapping \( \tau = z^2 + c \) of the complex plane;
- (b) directly from a dynamical problem in physics, as for instance a periodically delta-kicked particle;
- (c) from a reduction of the variables of a continuous system described by differential equations to a subspace, a Poincaré surface of section.

Here we will discuss the appealing case of a delta-kicked rotor with Hamiltonian

\[
H = \frac{p^2}{2} + \delta(t)K \cos q,
\]

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where the $\tau$-periodic comb function
\[
\delta_\tau(t) = \sum_{n=-\infty}^{+\infty} \delta(t/\tau - n) \quad (5)
\]
activates the potential only at a periodic sequence of delta-spike s. More generally such a kicked system can be written as
\[
H(p, q, t) = T(p) + \delta_\tau(t) V(q), \quad (6)
\]
which describes for example a particle of mass $m$ with kinetic energy $T(p)$ and coordinate $q$ which is kicked at periodic intervals $\tau$ with an impulse $-\tau V'(q)$ (the prime denotes differentiation with respect to the argument). Note that the strength of the kick depends on the position $q$. If $p$ and $q$ are the momentum and position immediately before a kick, the kick changes the momentum to $p = p - \tau V'(q)$, whereas the value of $q$ is fixed. After this kick, the motion is force-free and the momentum $p$ stays constant, whereas the coordinate $q$ increase with constant velocity $T'(p)$ during the time interval $\tau$ immediately before the next kick. The values at this time are therefore
\[
\begin{align*}
\bar{p} &= p - \tau V'(q) \\
\bar{q} &= q + \tau T'(\bar{p}).
\end{align*} \quad (7)
\]
Evaluating the Jacobian (3) for this mapping equation shows that the mapping is indeed area preserving.

For the well-known harmonic oscillator with kinetic energy $T(p) = p^2/2m$ and potential $V(q) = kq^2$, the kicked dynamics turns out to be linear, and hence non-chaotic:
\[
\begin{align*}
\bar{p} &= p - \tau kq \\
\bar{q} &= q + \tau p/2.
\end{align*} \quad (8)
\]
This is different, however, for a second system studied in almost all courses in elementary physics, namely the simple pendulum, i.e., a point mass attached to a fixed point by a string of fixed length in a homogeneous gravitational field. Unfortunately, the gravitational field can not be switched off easily, so that a kicked gravitational pendulum is only a gedankenexperiment. However, an equivalent system can be realized by an electric dipole with a fixed center rotating in a homogeneous electric field $E$ oriented in the z-direction. If $d$ is the dipole moment and $I$ the moment of inertia, the Hamiltonian of this system appears as
\[
H = \frac{p^2}{2I} + dE \cos q, \quad (9)
\]
where $q$ is the angle between the dipole and the field and $p$ is the angular momentum. Switching on the electric field periodically only during a very small time interval, we have an experimental realization of the kicked rotor in Eq. (4). Many other realizations are possible, as for instance using ultracold atoms in a modulated standing wave laser field [3–5]. These are, however, quantum systems and the experiments actually investigate the dynamics of a quantum kicked rotor, which deviates characteristically from the classical one discussed here. For more information on the kicked quantum rotor and quantum chaos in general see the book by Stöckmann [6]; for quantum chaos with cold atoms see the overview by Raizen [7].

Let us return now to our example of a classical kicked rotor (4). The rotor rotates with constant angular momentum between the kicks, where it changes suddenly by an amount $\tau dE \sin q$ depending on its momentary angle $q$. The motion from kick to kick is illustrated in Fig. 1.

Scaling the angular momentum as $\tau p/I \rightarrow p$, the mapping (4) for the kicked rotor reads
\[
\begin{align*}
\bar{p} &= p + K \sin q \\
\bar{q} &= q + \bar{p}
\end{align*} \quad (10)
\]
with a dimensionless parameter
\[
K = \tau^2 dE/I, \quad (11)
\]
which is a measure of the kick strength, proportional to the ratio between the potential energy of the field, $dE$, and the rotational energy for a rotation in resonance with the kicking period. This kicked rotor mapping is also known as the standard mapping or Chirikov mapping. It appears not only for the kicked rotor, but also in many other cases, for example if one approximates a general mapping locally in the vicinity of a fixed point (see [8, 9] for more details).

![FIG. 1: Dynamics of a dipole in a pulsed electric field $E$ in the z-direction, where the field amplitude is delta-pulsed $E \sim \delta_\tau(t)$ (compare Eq. (5)). Initially the rotor is oriented in direction $q_0$ and rotates with an angular momentum $p_0$. After a time $\tau$, the rotor has moved with constant angular velocity to a position $q_1$ and its angular momentum has jumped to $p_1$ because of the kick. This process continues generating a sequence $(q_n, p_n)$ of points in phase space.](image-url)
For a numerical study it is convenient to apply an additional scaling and replace \( p \) and \( q \) by \( 2\pi p \) and \( 2\pi q \). Then the mapping reads
\[
\begin{align*}
\bar{p} &= p + \frac{K}{2\pi} \sin 2\pi q \\
\bar{q} &= q + \bar{p},
\end{align*}
\] (12)
where the ‘angle’ \( q \) is taken modulo one in the interval \( 0 \leq q < 1 \). Note that the mapping depends only on a single parameter, the so called stochasticity \( K \).

III. INTERACTING WITH THE PROGRAM

Let us now study the kicked rotor dynamics numerically. This can be done, for example, using the program Kicked from the collection in [10] (see Sect. V), which allows to iterate general two-dimensional mappings and offers tools for displaying and analyzing the dynamics. This program is available on the EPAPS server (see [12]). It is controlled by a simple script language. In the beginning, the user can select from a number of predefined examples and change the code if desired and store the modified script files. Here we will not discuss the details of this script language, but instead illustrate its use by means of an example. The following script iterates the kicked rotor map (12):

```bash
# kicked rotor mapping
define
{K=1.0;k=K/2/M_PI;
 MOD=(x)->x-floor(x);}
variables q,p;
label("q","p");
range(0,-1.5,1,+1.5);
map
{ _p=p+k*sin(2*M_PI*q);
 _q=MOD(q+_p);}
ivp{p=-1.0;q=0.6;} iterate(5000);
```

This script first defines the parameter \( K = 1 \) of the kicked rotor mapping as well as the ‘modulo one’ function in terms of built-in functions. Then the names of the variables \( q, p \) are specified and used for labeling the axis in the range \( 0 < q < 1 \) and \( -1.5 < p < 1.5 \). The next two lines define the mapping and the following lines specify selected initial values which are iterated 5000 times.

By mouse-clicking the button Script (or alternatively pressing key <F9>) the script is successfully interpreted (the message success appears) and the iterated sequences \( (p_n, q_n) \), \( n = 0, \ldots, 5000 \) are displayed as black dots in a \( p, q \)-plane. Such a phase space diagram is denoted as a Poincaré plot and the window displayed on the screen as the Poincaré window, which is shown in Fig. 2.

Then the user can explore the mapping in more detail. In addition to modifying the script file, one can, for instance, select an additional initial point in phase space by mouse-clicking in the Poincaré window, which can then be iterated in a relatively slow mode by pressing the button Iterate or in a fast mode by activating the Loop box. Then the points of the iterated orbit are displayed in phase space. Of course, different colors can be chosen to distinguish different orbits, unpleasant orbits can be deleted and one can magnify details by zooming in. The generated graphics can be saved.

IV. COMPUTER EXPERIMENTS

In the following we will investigate the kicked rotor dynamics in some more detail starting from the Poincaré plot shown in Fig. 2. First we observe two different types of orbits: ‘regular’ ones, where the iterated points fill a curve, i.e., a one-dimensional subset of the plane, and ‘chaotic’ ones, where the iterated points apparently fill a two-dimensional region in phase space. These chaotic regions appear to be organized in bands with a distance \( \Delta p = 1 \). Three of these bands are visible in the figure. In fact, there exist infinitely many of such bands, which can be seen if the interval displayed on the \( p \)-axis is extended.

**Exploring phase space:** As a first experiment, we will ex-
explore the dynamics in more detail by iterating more trajectories by mouse-clicking of points of interest in phase-space. This eventually leads to a picture like Fig. 3 which shows additional regular orbits. Such an orbit always remains inside the band in which it was started. We will explore the properties of these orbits in some more detail below. If a trajectory is started in one of the chaotic bands on the other hand, one observes that it will – after quite a long time – visit also the other bands, i.e., the chaotic bands are connected. This can be demonstrated for a single trajectory started at \((p, q) = (-1.0, 0.9)\) in the lower chaotic band which is iterated over a long time \((10^8\) iterations) as shown in Fig. 4. This chaotic trajectory also explores the other bands, in fact it will reach arbitrary large momenta \(p\) for even longer times.

A simplified standard map: The standard mapping allows a further simplification, because trajectories started at \((p, q)\) and \((p + 1, q)\) lead to the same orbits up to a constant shift by \(\Delta p = 1\). One can therefore define a simplified mapping by taking \(p\) modulo one in the interval \(-0.5 \leq p < 0.5\). This yields an area preserving mapping of the unit square onto itself, where the opposite sides of the square are identified. This phase space has the topology of a torus and the mapping is an automorphism of this torus. In the program, this simplified mapping can be realized by changing the first mapping equation in the script to \(p = \text{MOD\_INT}(p + K \times \sin(2 \times \pi \times q))\) with a modulo-function definition \(\text{MOD\_INT}(x) = x - \lfloor x + 0.5 \rfloor\). Furthermore the displayed range should be adapted by \(\text{range}(0,-0.5,1,+0.5)\).

Exploring this simplified standard map with the computer yields for \(K = 1\) a picture as shown in Fig. 5. We observe an intricate web of islands embedded in an extended chaotic sea. The islands are organized by periodic orbits found in their centers. These periodic orbits have different periods \(N\) and are fixed points of the \(N\)-times iterated mapping \(M^N\), i.e., they return to their initial values after \(N\) applications of the mapping. More precisely, the island centers are stable fixed points, which means that initially close orbits will stay in their neighborhood for all times. Let us analyze this in more detail. One can immediately verify that the points \((p_0, q_0) = (0, 0)\) and \((0, 0.5)\) are mapped onto themselves under the mapping (12), they are fixed points of period one. The second one is clearly visible in the center of the big island, the other one is hidden in the chaotic region. We therefore expect the fixed point \((0, 0.5)\) to be stable and \((0, 0)\) to be unstable. Let us study the fate of nearby points mathematically. In their neighborhood, we can approximate the mapping by \(\sin 2\pi q \approx \pm (q - q_0)\) where the upper sign corresponds to \(q_0 = 0\) and the lower one to \(q_0 = 0.5\). This linearized map is described by the matrix

\[
L = \begin{pmatrix} 1 & \pm K \\ 1 & 1 \pm K \end{pmatrix}.
\]

Clearly, the linearized mapping \(M \approx L\) is also area conserving, \(\det L = 1\), with \(\text{trace } L = 2 \pm K\). The stability of the fixed point requires \(|\text{trace } L| = |2 \pm K| < 2\) (see, e.g., Ref. [10, Sect. 2.4.5]). Therefore, the first order fixed point at \(q_0 = 0\) is unstable, while the central fixed point at \(q_0 = 0.5\) is stable for \(K < 4\). Stable fixed points are also called elliptic, unstable ones hyperbolic because nearby points iterated by the lin-
FIG. 7: Simplified standard map for $K = 10$ showing well developed chaos. The chaotic region extends over the whole phase space.

earized map move along an ellipse or a hyperbola. A straightforward analysis shows that the ellipses around the stable fixed points are rotated by an angle $\chi$ with respect to the $q$-axis, where the angle $\chi$ is given by

$$\tan 2\chi = \frac{(L_{11} - L_{22})}{(L_{12} + L_{21})} = \frac{K}{(1 - K)}.$$ 

For the case $K = 1$, the central fixed point is therefore elliptic and the ellipses in the neighborhood should be tilted by an angle $\chi = \pi/4$, in agreement with the numerical results shown in Fig. 5.

Let us numerically study the loss of stability of the central fixed point for an increased kick strength close to $K = 4$. Figure 6 shows the phase plane for $K = 3.9$ and $K = 4.1$. Below the critical value $K = 4$, the central fixed point is still stable, however the elliptic orbits in its vicinity are highly elongated. For $K = 4.1$ the central elliptic fixed point has turned into a hyperbolic one and two new stable fixed points can be seen which are, however, of period two. This phenomenon is known as a bifurcation, more precisely a pitchfork bifurcation (see, e.g., [10, Sect. 2.4.4]).

We also observe a growth of the chaotic region. This process continues with increasing $K$. For strong fields as, e.g., for the case $K = 10$ shown in Fig. 7, we find well developed chaos, which means that the chaotic sea extends over the whole phase space. No obvious stability islands can be detected in this plot, but very tiny ones may still exist.

The weak kicking regime: It may also be of interest to have a brief look at the opposite regime of small values of $K$. As an example, Fig. 8 shows the dynamics for $K = 0.2$. Here all trajectories apparently look regular and the picture closely resembles the phase space diagram of a simple pendulum with Hamiltonian

$$H_0 = \frac{p^2}{2} + K \cos q.$$  

Here we have a stable fixed point with nearby oscillatory motion with small angular momentum. For large angular momentum we find a rotational motion extending over all angles. These two types of orbits are separated by a separatrix, a trajectory which passes through the unstable fixed point at $q = 0$.

Mathematically, the small $K$ limit can be studied using a Fourier identity for the periodic delta-function in the Hamiltonian, namely $\delta_c(t) = \sum_{m=-\infty}^{\infty} \delta^{2\pi m/\tau}$. For small $K$, the angle $q$ changes slowly. Neglecting the rapidly varying terms in the sum, i.e., keeping only the term with $m = 0$, we arrive at (14). More details concerning this analysis can be found in [9].

For $K = 0.2$, the dynamics is apparently regular. A closer look at the vicinity of the unstable fixed point, however, reveals a tiny chaotic region, which can be magnified by zooming in. Increasing $K$ to, e.g., $K = 0.4$, an extended chaotic region in the vicinity of the hyperbolic fixed point at $q = 0$ can be observed. The separatrix turns into a chaotic band. This chaotic band separates the inner oscillatory motion from the outer rotational region.

Suggestions for further studies:

(a) Study the mapping equations for a kicked harmonic oscillator (8) numerically in order to see the remarkably simple behavior of a fully regular system.

(b) Figure 5 shows a number of island chains. Find out, which of these islands are connected by common trajectories hopping from one island to the next. Study the periods of these orbits.

(c) Locate period-two fixed points and explore their stability and bifurcation properties, both numerically and analytically.

(d) Study the behavior of the kicked rotor in the vicinity of a stability island by zooming-in. Repeat this procedure to observe that the highly organized structure continues up to arbitrary small scales in a self-similar manner.
(e) For small values of $K$, the chaotic regions are confined to separated bands. Explore the transition to global stochastic motion in the region $0.9 < K < 1.0$. Find numerically the critical value $K_c$ where the last regular orbit extending over all angles is destroyed and compare your result with the value $K_c = 0.9716$ given in the literature [9].

(f) Above the critical parameter $K_c$, the kicked rotor iteration (12) for initial values in the chaotic sea becomes diffusive. This diffusive growth is suppressed, however, for a quantum kicked rotor; see, e.g., the book by Stöckmann [6] for more information about this quantum suppression of classical chaos.

V. IMPLICATIONS IN TEACHING

The kicked rotor is a simple system and can be quite easily used in teaching chaotic dynamics. The basic mapping can be easily derived, even without the use of delta functions. Moreover, an experimental realization of such a system can be imagined. The basic insight originating from a more detailed study of the kicked rotor is the fact, that one cannot judge the properties of a nonlinear system from the mathematically simple looking mapping formula.

This requires, however, a numerical iteration of the mapping equations. For a small number of iterations, this can be carried out by hand using a pocket calculator. But even in this case, the basic features of the dynamics can only be imagined by a graphical plot of the iterated points in phase space. Therefore the use of a computer is unavoidable in studying or teaching chaotic dynamics.

About fifteen years ago two of the authors therefore published a collection of PC programs [11] in order to facilitate teaching chaotic dynamics even for people with no or very limited computational background. The programs provide an easy to use platform to explore selected dynamical systems relevant to physics. Caused by the rapid development of the computational world, these programs written in Turbo Pascal appeared in an old-fashioned design compared to the up-to-date standard. Even more important, those programs would not run properly under recent versions of the Windows operating system. Therefore all programs of the new edition [10] have been entirely rewritten in C++ and, of course, revised and polished and can be used under Windows or Linux operating systems. As an example, the program KICKED discussed above is available on the EPAPS server (see [12]).

The general structure of the presentation is, however, not changed. Each chapter starts with an outline of the theoretical background, a brief discussion of the numerical techniques and a description of the interaction with the program. The main parts are the computer experiments described in detail. These experiments can be repeated by the reader starting with preset settings of the program and getting more and more familiar with the possible actions and modifications. The programs are flexible and additional investigations are suggested and desired. At the end of each chapter, the computer studies are extended by a discussion of real experiments and empirical evidence, as well as references to the literature.

In addition to the kicked rotor, many other systems can be investigated in this way, as for example billiard systems, the double pendulum, chaotic scattering, Fermi acceleration, the Duffing oscillator, nonlinear electronic circuits and general discrete mappings, which may help to illuminate the fascinating dynamics of chaotic systems.

[12] A compiled and ready to run version of the Kicked program is made available on the EPAPS server. See Document No. xxx. This document can be reached through a direct link in the online article’s HTML reference section or via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html).