Bose-Hubbard dimers, Viviani’s windows and pendulum dynamics

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Abstract. The two-mode Bose-Hubbard model in the mean-field approximation is revisited emphasizing a geometric interpretation where the system orbits appear as intersection curves of a (Bloch) sphere and a cylinder oriented parallel to the mode axis, which provide a generalization of Viviani’s curve studied already in 1692. In addition, the dynamics is shown to agree with the simple mathematical pendulum. The areas enclosed by the generalized Viviani curves, the action integrals, which can be used to semiclassically quantize the \( N \)-particle eigenstates, are evaluated. Furthermore the significance of the original Viviani curve for the quantum system is demonstrated.

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1. Introduction

The two-mode Bose-Hubbard system is a popular model in multi-particle quantum theory. It describes \( N \) bosons, hopping between two sites with on-site interaction or a spinning particle with angular momentum \( N/2 \). Despite its simplicity, it offers a plethora of phenomena and applications motivating an increasing number of investigations, somewhat similar to the harmonic oscillator in single-particle quantum mechanics. In the limit of large \( N \) it can be described by a non-linear Schrödinger or Gross-Pitaevskii equation which generates a classical Hamiltonian dynamics (see, e.g., [1–6] for some studies related to the following). Moreover it has been shown recently [7–12] that a semiclassical WKB-type construction can be used to approximately recover quantum effects, such as the eigenstates and interference effects for finite (and even small) numbers of particles. This is based on classical action integrals, as usual in semiclassics.

In the present note, we point out an interesting relation between these contemporary studies and much older ones by astronomers and mathematicians from the times of Galilei or even earlier in ancient Greece as, e.g., Vincenzo Viviani (1622–1703) [13] or Eudoxus of Cnidus (408 BC–355 BC) [14]. The basic link between these studies is the
family of closed curves on the surface of a sphere and the surface area they enclose, appearing as phase space action integrals in the Bose-Hubbard dimer.

The paper is organized as follows: In section 2 we give a brief review of the background of Viviani curves, and introduce a generalization. We furthermore calculate the areas enclosed by the generalized Viviani curves. In section 3 we describe the Bose-Hubbard dimer and its mean-field approximation and show that the dynamical trajectories of the latter are given by generalized Viviani curves. We also show how the dynamics of the mean-field system can be directly related to that of a mathematical pendulum. Finally we analyze the relevance of the Viviani curve for the quantum Bose-Hubbard dimer. We conclude with a summary in section 4.

2. Viviani curves

In 1692 Vincenzo Viviani, a disciple of Galileo Galilei, proposed a problem of the construction of four equal windows cut out of a hemispherical cupola such that the remaining surface area can be exactly squared [13] (see also [15] and references therein). The solution to this problem is given by an intersection of the hemisphere and a cylinder whose diameter equals the radius of the hemisphere. This intersection curve of a sphere and a cylinder tangent to the diameter of the sphere and its equator is the famous Viviani curve, which can be generalized to an intersection with an arbitrary cylinder. For a recent extended study of such curves see [16,17].

2.1. (Generalized) Viviani curves

Consider a cylinder of radius \( r \) centered around an axis in the \( z \)-direction shifted by \( a \) in the \( x \)-direction:

\[ (x-a)^2 + y^2 = r^2. \]  

We will parametrize the basis of the cylinder by an angle \( \phi \),

\[ x = a + r \cos \phi , \quad y = r \sin \phi \]  

as shown in figure 1.

The generalized Viviani curves are given by the intersection curves of this cylinder with the unit sphere centered at the origin illustrated in figures 2 and 3. The radius \( r \) of the cylinder varies between \( r = 0 \) and \( r = 1 + a \), otherwise there is no intersection with the sphere. For \( r = 1 + a \) the cylinder touches the sphere at \( x = -1 \). We have to distinguish two cases (see figure 1):

(i) For \( 0 < a < 1 \) and \( 0 < r < 1 - a \) all points on the circle lead to a full intersection of the cylinder and the sphere. We obtain two single intersection loops on the northern or southern hemisphere containing the points \((a,0,+\sqrt{1-a^2})\) and \((a,0,-\sqrt{1-a^2})\), respectively, as illustrated on the left of figures 2 and 3.
Figure 1. Projection of the intersection of the unit sphere and the cylinder on the 
\((x,y)\)-plane for \(a = 0.25, r = 0.6\) (left) and \(a = 1.25, r = 0.8\) (right). This corresponds 
to the dynamical trajectories of the mean-field Bose-Hubbard dynamics, where the 
dashed part of the circle is energetically inaccessible.

(ii) For \(0 < a < 1\) and \(1 - a < r < 1 + a\) as well as for \(a > 1\) and \(a - 1 < r < 1 + a\) 
there is no intersection of the cylinder and the sphere for \(|\phi| < \phi_0\) with

\[
\phi_0 = \arccos \frac{1 - a^2 - r^2}{2ar}.
\]  

The curve of intersection with the sphere is a single loop extending from the 
northern to the southern hemisphere (see figure 2 and 3 (right)).

Three special situations are of interest (see figure 4):

(a) For \(r = a\) the cylinder passes through the center of the sphere and the intersection 
loop passes through the north- and south pole. As we shall see later, for the Bose-
Hubbard dimer, this is the frequently considered situation where the system is 
initially prepared in the lower or upper state.

Figure 2. Intersection of the unit sphere and a cylinder displaced by \(a = 0.5\) along 
the \(x\)-axis for radius \(r = 0.4, 0.5\) and \(0.6\) (from left to right). The figure in the middle 
is the Viviani curve.
Figure 3. Intersection of the unit sphere and a cylinder displaced by $a = 0.7$ along the $x$-axis for radius $r = 0.2, 0.3$ and $0.4$ (from left to right).

(b) For $r = 1 - a$, which is only possible for $a \leq 1$, the cylinder is tangent to the sphere at the point $s_{0+} = (1, 0, 0)$. This implies that the intersection curve is a figure-eight loop with a self-intersection at $s_{0+}$ (see figure 3 (middle)). This curve is also known as the Hippopede of the Greek astronomer and mathematician Eudoxus of Cnidus (408 BC – 355 BC) [14, 18]. In the Bose-Hubbard mean-field dynamics it appears as a separatrix curve, which separates the flow inside (for smaller values of $r$) from the flow outside (for larger values of $r$).

(c) In the most singular situation cases (a) and (b) coincide, i.e. the cylinder passes through the center and touches the sphere. This happens only for $r = a = \frac{1}{2}$ and is the original Viviani case illustrated in figure 2 (middle). Using the parametrization (2) the Viviani curve is given as

$$
x = \frac{1}{2} + \frac{1}{2} \cos \phi, \quad y = \frac{1}{2} \sin \phi$$

and $z^2 = 1 - x^2 - y^2 = (1 - \cos \phi)/2 = \cos^2(\phi/2)$ and therefore $z = \pm \cos(\phi/2)$. This implies the relation $\varphi = \phi/2$ between the azimuthal polar angle $\varphi$ and the

Figure 4. Projection of the intersection of the sphere with a cylinder on the $(x,y)$-plane for the special cases $r = a$ (case (a)), $r = 1 - a$ (case (b)) and $r = a = 1/2$ (Viviani case (c)).
angle $\phi$, which can also be found by purely geometric arguments. Therefore the Viviani curve is given by

$$\begin{align*}
(x, y, z) &= (\cos^2 \varphi, \cos \varphi \sin \varphi, \sin \varphi).
\end{align*}$$

Comparing with spherical polar coordinates (see (32) below) we observe that the Viviani curve (5) can also be defined by the simple condition that the azimuthal angle is equal to the polar angle measured from the equator:

$$\varphi = \pi/2 - \vartheta.$$  \hfill (6)

In spite of the fact that only curve (5) is the one originally described by Viviani [13], we will deliberately denote all the intersection curves of a cylinder and a sphere as (generalized) Viviani curves. Alternatively these curves are known as euclidean spherical ellipses. This is due to their remarkable property that the sum of the euclidean distances to two focal points $x_F = a/(1+r)$, $y_F = 0$, $z_F = \pm \sqrt{1 - x_F^2}$ on the sphere equals a constant $2c$ with $c^2 = z_F^2 a/x_F$ [16]. These euclidean spherical ellipses must be distinguished from the more popular version, where the distance is measured by the arc length on the surface. The latter version found many applications in navigation. Figure 5 shows the original Viviani curve (5), whose focal points are at $1/3, 0, \pm \sqrt{8}/3$, as well as two generalized Viviani curves, i.e. euclidean spherical ellipses, which are chosen to possess the same focal points, however with $c$ smaller or larger than the value $c = 2/\sqrt{3}$ for the Viviani curve. Let us point out again that these euclidean spherical ellipses can appear as two disconnected loops.

Note also that the projection of the generalized Viviani curves on the $(y, z)$-plane, the curves

$$\begin{align*}
(z^2 - 1 + a^2 - r^2)^2 + 4a^2 y^2 &= 4a^2 r^2,
\end{align*}$$

Figure 5. Viviani curve (red) and focal points (red dots) together with two confocal euclidean spherical ellipses (black). Also show are the projections on the $(x, y)$-plane.
are well-known functions, at least for $r = a$, where the cylinder passes through the midpoint of the sphere (case (a)), they are known as Cassini ovals and for $r = 1 - a$, where the cylinder is tangent to the sphere, one obtains an eight-curve, also known as Lemniscate of Gerono.

The previous considerations exclusively discussed the geometry of the intersection curves. These curves can, however, be also generated by the trajectories of a time evolution. For the (original) Viviani curve, such a dynamical generation is often realized on the basis of eq. (6) by simply assuming a combined rotation along the circle of latitude, $\vartheta = \pi/2 - \omega t$, and the meridian, $\varphi = \omega t$. Also the more general separatrix (b), the Hippopede of Eudoxus, was constructed in an effort to explain the retrograde motion of the planets by a rotation of nested spheres that share a common center with the same frequency around different axes [14]. As we shall see later, the same curves arise as the dynamical trajectories of the mean-field approximation for the Bose-Hubbard dimer.

2.2. Area integrals

The area enclosed by a generalized Viviani curve, or more precisely the sum of the enclosed areas if this curve consists of two loops, is of particular interest. Historically, of course, because this is the origin of the old pseudo-architectural problem posed and solved by Vincenzo Viviani in 1692, as stated already at the beginning of this section. Today, in the context of the Bose-Hubbard dimer discussed in section 3, the area is the basis for a semiclassical quantization of the eigenvalues of the $N$-particle Bose-Hubbard dimer [7,8,12].

The calculation of an area $S$ enclosed by a curve on the sphere can be conveniently carried out by means of the area conserving projection of the unit sphere onto a cylinder touching the sphere along the equator, which was already known by Archimedes:

$$ (x, y, z) \rightarrow (x', y', z') = (\cos \varphi, \sin \varphi, z), $$

that is,

$$ \varphi = \arctan \frac{y}{x} \quad \text{for} \quad z \neq \pm 1. $$

The poles $(0, 0, \pm 1)$ are projected onto circles $(\cos \varphi, \sin \varphi, \pm 1)$. The area element is $dS = z(\varphi) \, d\varphi$, where $z(\varphi)$ is the $z$-component of the points $(x, y, z)$ on the curve parametrized by the azimuthal angle $\varphi$.

As a first example we calculate the area on the sphere enclosed by the Viviani curve. The full area enclosed by the figure-eight-shaped Viviani curve is given by

$$ S_V = 2\pi - 2S_0, $$

where $S_0$ is the area between one half of the Viviani curve and the equator. From the $z$-component given in (5) we see that the cylinder projection (8) maps the Viviani curve onto a sine-function and thus we find

$$ S_0 = 2 \int_0^{\pi/2} z(\varphi) \, d\varphi = 2 \int_0^{\pi/2} \sin \varphi \, d\varphi = 2. $$
Note that on a more general sphere with radius $R$ this area is given by $4R^2$ as required from the solution to the original Viviani problem. The area enclosed by the Viviani curve is thus given by

$$S_V = 2\pi - 2S_0 = 2\pi - 4 \approx 2.2832. \quad (12)$$

For evaluating the area integral in the general case, it is convenient to transform to the variable $\phi$ by means of (2):

$$z(\varphi) \, d\varphi = z(\phi) \, d\phi = r \sqrt{1 - r^2 - a^2 - 2ar \cos \phi} \, \frac{r + a \cos \phi}{a^2 + r^2 + 2ar \cos \phi} \, d\phi, \quad (13)$$

and the area outside the curve is given by

$$S_0 = r \int_{\phi_0}^{\pi} \sqrt{1 - r^2 - a^2 - 2ar \cos \phi} \, \frac{r + a \cos \phi}{a^2 + r^2 + 2ar \cos \phi} \, d\phi, \quad (14)$$

where the lower bound $\phi_0$ is equal to zero for case (i) and otherwise given by (3). The integrand is proportional to $1/(r-a)$ for $\phi = \pi$ so that we have an integrable singularity for $r = a$. The area $S$ inside the curve is then equal to zero for $a > 1$, $r < a - 1$ and equal to $4\pi$ for $r > 1 + a$, otherwise it is given by

$$S = \begin{cases} 
-4S_0 & \text{for } r < a \\
2\pi - 2S_0 & \text{for } r = a \\
4\pi - 4S_0 & \text{for } r > a.
\end{cases} \quad (15)$$

For the special cases distinguished above the area integral can be evaluated in closed form:

(a) For $r = a$, the transformation from $\varphi$ to $\phi$ simplifies to $\varphi = \phi/2$ and the integral to

$$S_0 = \int_{\phi_0}^{\pi} \sqrt{1 - 2a^2 - 2a^2 \cos \phi} \, d\phi$$

$$= \begin{cases} 
2E(4a^2) & \text{for } a \leq 1/2 \\
4a\left(E(1/4a^2) - (1 - 1/4a^2)K(1/4a^2)\right) & \text{for } a > 1/2
\end{cases} \quad (16)$$

where $E(m)$ and $K(m)$ are complete elliptic integrals of the first and second kind with parameter $m$.

(b) For $r = 1 - a$ (only possible for $a < 1$, special case (b) mentioned above) where the intersection curve is an eight-shaped curve with a double-point at the fixed point $s_{0+} = (1, 0, 0)$ where sphere and cylinder touch each other, the area integral can be also calculated in closed form with the amazingly simple result [17, sect. 3.2]

$$S = 8 \arcsin \sqrt{1-a} - 8 \sqrt{a(1-a)} \quad (17)$$

(c) For the Viviani case $r = a = 1/2$ both results agree with $S_V$ in (12).

As an example, figure 6 shows on the left the area $S$ enclosed by the (generalized) Viviani loops as a function of the radius $r$ of the cylinder for three values of the displacement $a$ chosen in the different regions. In each case, the area increases
Figure 6. Left: Area $S/(4\pi)$ as a function of cylinder radius $r$ for $a = 0.25$ (solid blue), $a = 0.5$ (dashed red) and $a = 1.5$ (dashed dotted black). Right: Area $A = S/(4\pi)$ as a function of the inverse cylinder displacement $a^{-1}$ for the critical curve passing through the poles (solid blue) and for the critical figure-eight-shaped curve passing through the point $s_{0+} = (1, 0, 0)$ (dashed red), i.e. eq.(17). The black dashed dotted curve shows the related time period $T/(4\pi)$ of the mean-field Bose-Hubbard dynamics for $v = 1$ (see eq. (47)).

monotonically from 0 to $4\pi$. The plot on the right shows the area as a function of the inverse cylinder displacement $a^{-1}$ for the curves (a) passing through the poles ($r = a$). These pole trajectories exist for all values of $a$. In the limit $a^{-1} \to 0$ the cylinder with $r = a$ approaches the $(y, z)$-plane and intersects the sphere in a great circle, which divides it into two equal hemispheres of area $2\pi$. With increasing $a^{-1}$ the intersection loop is deformed and the area enclosed shrinks, but it is still a single closed curve up to the critical point $a = 1/2$, the Viviani case, where it bifurcates into two loops encircling the two extrema.

Also shown in this figure is the area integral for case (b), see eq. (12). These figure-eight shaped trajectories only exist for $a < 1$ and, with decreasing $a$, the area enclosed grows. For $a = 1/2$ the cases (a) and (b) coincide, and both curves pass through the Viviani area $S/4\pi \approx 0.1817$ according to (12). For still smaller values of $a$ the cylinder center approaches the center of the sphere and the area enclosed by the loops (a) through the poles goes to zero as $S \approx 2\pi a^2$ whereas the area enclosed by the loops through the touching point approaches the full surface $4\pi$.

We shall return to discussing the role of the area integrals for the quantum spectrum of the Bose-Hubbard dimer in section 3.2 after reviewing the quantum Hamiltonian and its mean-field approximation in the following.

3. The Bose-Hubbard dimer and the mean-field approximation

The two-mode Bose-Hubbard system, describing cold bosonic atoms on a ‘lattice’ consisting of only two sites, is a standard model in the field of cold atoms [1–3]. It
is described by the Hamiltonian

\[ \hat{H} = \epsilon (\hat{n}_1 - \hat{n}_2) + v (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{1}{2} c (\hat{n}_1 - \hat{n}_2)^2, \]  

with mode energies \( \pm \epsilon \), coupling \( v \) and on-site interaction \( c \). The \( \hat{a}_j \) and \( \hat{a}_j^\dagger \) denote particle annihilation and creation operators in mode \( j \) respectively. The total particle number \( \hat{N} = \hat{n}_1 + \hat{n}_2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \) is a conserved quantity. Despite its simple theoretical structure the Bose-Hubbard dimer is of considerable importance also from an experimental point of view, describing for example ultracold atoms in a double-well trap or in the ground state of an external trap with two internal degrees of freedom \[19,20\].

Introducing self-adjoint angular momentum operators \( \hat{L}_x, \hat{L}_y \) and \( \hat{L}_z \) according to

\[ \hat{L}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \]
\[ \hat{L}_y = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \]
\[ \hat{L}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), \]

with \( SU(2) \) commutation relation \([\hat{L}_x, \hat{L}_y] = i \hat{L}_z \) and cyclic permutations, the Hamiltonian \(18\) can be written as:

\[ \hat{H} = 2\epsilon \hat{L}_z + 2v \hat{L}_x + 2c \hat{L}_z^2. \]

Here the conservation of the particle number \( N \) appears as the conservation of \( L^2 = \frac{N}{2}(N^2 + 1) \), i.e. the rotational quantum number \( L = N/2 \). Note that the Hamiltonian \(20\) is also known as the Meshkov-Lipkin-Glick Hamiltonian, an angular momentum model originally introduced in the context of nuclear physics as a solvable model against which to check typical approximation schemes of many-particle physics \[21–23\]. The mean-field approximation in the context of cold atoms is closely related to the classical approximation for the Lipkin-Meshkov-Glick system, which has been the subject of many studies in applications to quantum information theory or quantum phase transitions in the past (see, e.g., \[24–27\] and references therein).

Both the static and dynamic properties of the full many-particle system \(18\) can be numerically deduced in a straightforward manner. This makes it an ideal model for the investigation of the correspondence to the approximate mean-field description, which we shall briefly discuss in the following.

### 3.1. Mean-field dynamics

The mean-field approximation can be formally obtained by replacing the bosonic operators by c-numbers: \( \hat{a}_j \rightarrow \psi_j, \hat{a}_j^\dagger \rightarrow \psi_j^* \). Taking the macroscopic limit \( N \rightarrow \infty \) with \( cN = g = \text{const.} \) yields the discrete nonlinear Schrödinger or Gross-Pitaevskii equation

\[ i \dot{\psi}_1 = (\epsilon + g(|\psi_1|^2 - |\psi_2|^2)) \psi_1 + v \psi_2 \]
\[ i \dot{\psi}_2 = v \psi_1 - (\epsilon + g(|\psi_1|^2 - |\psi_2|^2)) \psi_2, \]

for \( \hbar = 1 \), where \( g = cN \) is the overall interaction between the particles.
Similar to linear time-dependent Schrödinger equations also the nonlinear dynamics\(^{21}\) possesses a canonical structure\(^{28,29}\): It can be derived from a classical Hamiltonian function
\[
H_{\text{cl}} = \epsilon(|\psi_1|^2 - |\psi_2|^2) + v(\psi_1^*\psi_2 + \psi_1\psi_2^*) + \frac{g}{2}(|\psi_1|^2 - |\psi_2|^2)^2,
\]
(22)
with canonical equations
\[
i\dot{\psi}_j = \frac{\partial H_{\text{cl}}}{\partial \psi_j^*}.
\]

One can also formulate the nonlinear two-mode system in terms of the Bloch vector of a spin-\(\frac{1}{2}\) system, introducing the spin components
\[
s_x = \frac{1}{2}(\psi_1^*\psi_2 + \psi_1\psi_2^*),
\]
\[
s_y = \frac{1}{2}(\psi_1^*\psi_2 - \psi_1\psi_2^*),
\]
\[
s_z = \frac{1}{2}(\psi_1^* - \psi_2\psi_2^*)
\]
in accordance with \((19)\). In these variables the total energy takes the form
\[
H_{\text{cl}} = 2\epsilon s_z + 2vs_x + 2gs_z^2,
\]
(24)
and the nonlinear Schrödinger equation \((21)\) translates to nonlinear Bloch equations of the form
\[
\dot{s}_x = -2\epsilon s_y - 4gs_y s_z
\]
\[
\dot{s}_y = 2\epsilon s_x + 4gs_x s_z - 2vs_z
\]
\[
\dot{s}_z = 2vs_y
\]
(25)
These equations conserve the normalization, that is, the motion of the spin vector \(s = (s_x, s_y, s_z)\) is confined to the surface of the Bloch sphere with radius \(|s| = 1/2\).

Alternatively one can investigate the system in terms of the coordinates \(p = |\psi_1|^2 - |\psi_2|^2\) and \(q = (\arg(\psi_2) - \arg(\psi_1))/2\), which are related to the coordinates on the Bloch sphere via
\[
p = 2s_z, \quad 2q = \arctan s_y/s_x.
\]
(26)
That is, they are similar to the (area preserving) projection of the vector \(s\) on a cylinder touching the sphere at the equator as introduced before in equations \((8)\) and \((9)\).

The classical energy, that is, the Hamiltonian function, in terms of \(p\) and \(q\) reads
\[
H_{\text{cl}} = \epsilon p + v\sqrt{1 - p^2}\cos(2q) + \frac{g}{2}p^2,
\]
(27)
and the equations of motion \((21)\)
\[
\dot{p} = 2v\sqrt{1 - p^2}\sin(2q)
\]
\[
\dot{q} = \epsilon - v\frac{p}{\sqrt{1 - p^2}}\cos(2q) + gp,
\]
(28)
(29)
are again canonical, that is, \(\dot{q} = \partial H_{\text{cl}}/\partial p, \dot{p} = -\partial H_{\text{cl}}/\partial q\). This describes the motion of a pendulum whose length, however, depends dynamically on the momentum \(p\). The energy \(E = H_{\text{cl}}\) is conserved.

In the following we will confine ourselves to the case of a symmetric Bose-Hubbard dimer, \(\epsilon = 0\), which captures important characteristics of the dynamics. In addition we
will assume \( v > 0 \) without loss of generality. In this case, the fixed points of the dynamics of the Bloch vector \((26)\), corresponding to stationary solutions of the Gross-Pitaevskii equation, are given by

\[
s_y = 0 , \quad s_z \in \left\{ 0, 0, \pm \frac{1}{2} \sqrt{1 - v^2/g^2} \right\}. \tag{30}
\]

Thus we have two fixed points for \(|g| \leq v\) with energies \( E = -v \) and \( E = v \) which are a minimum and a maximum of the energy surface. If the interaction strength \(|g|\) is increased one of these extrema bifurcates at the critical interaction \(|g| = v\) into a saddle point, still at the equator, and two extrema located away from the equator at 

\[
s_z = \pm \frac{1}{2} \sqrt{1 - v^2/g^2} \quad \text{with energy } E_m = \frac{g}{2} (1 + v^2/g^2),
\]

two maxima for repulsive interaction \( g > 0 \) or two minima for attractive interaction. In both cases the other extremum (minimum with energy \( E = -v \) for \( g > 0 \) or maximum with energy \( E = v \) for \( g < 0 \)) remains at the equator. In this supercritical case the two fixed points \( s_z \neq 0 \) correspond to stationary states mainly populating one of the levels, the self trapping states.

For the following discussion it will be convenient to rescale the spin components as 

\[
x = 2s_x, \quad y = 2s_y, \quad z = 2s_z
\]

where the nonlinear Bloch equations \((26)\) in the symmetric case,

\[
\begin{align*}
\dot{x} &= -2gyz \\
\dot{y} &= +2gxyz - 2vz \\
\dot{z} &= 2vy,
\end{align*}
\]

restrict the motion of the vector \( s = (x, y, z) \) to the unit sphere \(|s|^2 = x^2 + y^2 + z^2 = 1\).

We will assume that the parameters \( v \) and \( g \) are non-negative, otherwise the dynamics can be obtained by simple symmetry arguments. We will also use spherical polar coordinates

\[
(x, y, z) = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta).
\]

Of basic importance for the structure of the flow on the sphere are the fixed points

\[
s_{0\pm} = (\pm 1, 0, 0) \quad \text{and} \quad s_{1\pm} = (a, 0, \pm \sqrt{1 - a^2}) \quad \text{for} \quad a^2 < 1
\]

with \( a = v/g \). The energy

\[
E = vx + \frac{1}{2} g z^2
\]

(see eq. \((24)\)) is conserved and the fixed points \( s_{0\pm} \) correspond to a maximum and a minimum energy \( E_{0\pm} = \pm v \). At \( a = 1 \) the maximum bifurcates into a saddle point at \( s_{0+} \) and two maxima at \( s_{1\pm} \) with energy \( E_{1\pm} = g(1 + a^2)/2 \) in the self trapping transition.

It is now easy to see that the dynamical trajectories \((31)\) are exactly given by the generalized Viviani curves introduced in section \(2.1\). Inserting \( z^2 = 1 - x^2 - y^2 \) we can rewrite equation \((34)\) in the form \((1)\) with

\[
r = \sqrt{1 + a^2 - 2E/g}.
\]

\[
\]
Thus, the trajectories generated by the flow (31) can be interpreted as curves of intersection of the unit sphere with a cylinder as illustrated in figures 2 and 3, that is, Viviani curves.

Of special interest for the two-mode Bose-Hubbard dynamics is the case where initially the system is in the lower or upper state, i.e., at one of the poles on the Bloch sphere. This imposes the condition \( r = a \). For \( a > \frac{1}{2} \) (i.e. \( g < v^2/2 \)) this trajectory traces out a closed euclidic ellipse on the sphere. For \( a = \infty \) this is a great circle in the \((y,z)\)-plane which tightens if \( a \) is reduced. Note that the intersection points of this ellipse with the equator at \( x = 1/2a, y = \pm \sqrt{1-1/(4a^2)} \) approach each other until they meet for \( a = \frac{1}{2} \) at the fixed point \( s_+ = (1,0,0) \), a double point of the figure eight shaped trajectory. For \( a < \frac{1}{2} \) the ellipse consists of two loops on the northern and southern hemisphere which end up as circles around the poles for \( a = 0 \).

Let us now return to the time dependence generated by the flow (31). The motion is periodic with a period \( T \) derived by several authors before (see, e.g., [12,30,31]). Here we present a very simple derivation, which also provides new insight into the dynamics.

For the simplest case, \( g = 0 \), the flow equations (31) are linear. The \( x \)-component is conserved, \( x = x_0 \), and we find a global rotation around the \( x \)-axis with frequency \( \omega_x = 2v \). In our picture this corresponds to the limit \( a, r \to \infty \) with fixed \( a - r = x_0 \). Somewhat more interesting is the case \( v = 0 \). Here the \( z \)-component is conserved, \( z = z_0 \), and we find a rotation around the \( z \)-axis. Here, however, we have

\[
\dot{x} = -2gz_0y, \quad \dot{y} = 2gz_0x \quad \Rightarrow \quad \ddot{x} = -2gz_0 \dot{y} = -4g^2z_0^2 x
\]

and the frequency

\[
\omega_z = 2g|z_0| = 2g\sqrt{1-r^2}
\]

varies from a maximum \( \omega = 2g \) at the poles to \( \omega = 0 \) at the equator.

For the general case, inserting the time derivative of transformation (2) into the flow equations (31) we find

\[
\dot{\phi} = 2gz = \pm 2g\sqrt{1-a^2-r^2-2ar \cos \phi},
\]

which is just the angular velocity of a simple mathematical pendulum. This is even more evident from the second time derivative:

\[
\ddot{\phi} = 2g\ddot{z} = 4vgy = 4vgr \sin \phi
\]

or with \( \phi = \pi + \theta \)

\[
\ddot{\theta} + 4vgr \sin \theta = 0,
\]

which is the equation of motion for a mathematical pendulum, where the ratio between gravitational acceleration and pendulum length is replaced by \( 4vgr \), which is an energy dependent constant. The two cases (i) and (ii) distinguished in section 2.1 are simply the well-known librational and rotational motions of the pendulum.

(i) Rotation: With the abbreviations

\[
b = \frac{a^2 + r^2 - 1}{2ar}, \quad B = 2g\sqrt{2ar}.
\]
we obtain the period as twice the integral over $\dot{\theta}^{-1}$ from 0 to $\theta_0 = \pi$ as

$$T_R = \frac{1}{B} \int_0^\pi \frac{d\theta}{\sqrt{\cos \theta - b}} = \frac{1}{g \sqrt{m \ar} K(1/m)},$$

(note that $|b| > 1$) with

$$m = \frac{1 - b}{2} = \frac{1 - (r - a)^2}{4ar} \geq 1$$

and $K$ is the complete elliptic integral of the first kind.

(ii) *Libration:* For $|b| \leq 1$ we have a hindered rotation restricted to the interval $|\theta| \leq \theta_0 = \arccos b$, and the period is 4 times the integral over the interval $0 < \theta < \theta_0$, that is,

$$T_L = \frac{4}{B} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - b}} = \frac{2}{g \sqrt{m \ar} K(m)}.$$

where the parameter (43) satisfies $|m| \leq 1$.

This, as well as the explicit construction of the solution in terms of Jacobi elliptic functions, is, of course, well known for the pendulum and also for the Bose-Hubbard dimer [31] (see, e.g., [32, 33]). It should be noted, that also the integrals of section 2.2 appear as action integrals for the pendulum. Some limiting cases may be of interest:

(1) For trajectories with $r$ close to the boundary of the allowed dynamical region, i.e. for $r \gtrsim a - 1$ for $a > 1$ or for $r \lesssim a + 1$, we have $m \approx 0$ and with $K(0) = \pi/2$ the period (44) reduces to

$$T_{L\pm} = \frac{\pi}{g \sqrt{a(a \mp 1)}} = \frac{\pi}{g \sqrt{v(v \mp g)}},$$

where one recognizes the celebrated Bogoliubov frequency $\Omega = 2\pi/T = 2\sqrt{v(v \mp g)}$ for small excitations around the ground- and highest excited state, respectively [34, 35].

Clearly for this small angle oscillation this result can be directly obtained from the pendulum equation (40). On the Bloch sphere this is an oscillation in the vicinity of the fixed points $s_{0\pm} = (\pm 1, 0, 0)$ (compare eq. (33)).

(2) For $a < 1$ and $r = 0$ we find the two fixed points $s_{1\pm} = (a, 0, \pm \sqrt{1 - a^2})$ from (33). The oscillation period in their vicinity can be found from (39) as

$$T_R = 2g \sqrt{1 - a^2} = 2\sqrt{g^2 - v^2}.$$

(3) In the special case (b) mentioned in section 2.1 we have $a < 1$ and $r = 1 - a$ (i.e. $b = -1$ and $m = 1$) the orbit approaches the unstable fixed point $s_{0+} = (1, 0, 0)$ along the separatrix and the period becomes infinite.

(4) For $r = a$ the orbit passes through the poles (case (a) in section 2.1) and the period is given by

$$T = \begin{cases} 
\frac{2}{g} K(4a^2) & \text{for } a < \frac{1}{2} \\
\frac{1}{g} K(1/4a^2) & \text{for } a > \frac{1}{2} 
\end{cases}$$

as shown in figure 6.
We have shown that the restriction to the cylinder \([1]\) reduces the dynamics of the symmetric dimer on the Bloch sphere to a simple pendulum motion on the level sets of constant energy. A similar construction for Euler’s equations for the free rigid body can be found in \([36,37]\).

### 3.2. Full many-particle description

In this section we will illustrate some implications of the properties of the mean-field approximation discussed above for the many-particle two-mode Bose-Hubbard Hamiltonian \([18]\) with \(\hbar = 1\), where we confine ourselves to the supercritical case \(g > v\) \((v > 0, g > 0)\).

Diagonalizing the \(N\)-particle Hamiltonian we obtain the \(N + 1\) energy eigenvalues \(E_n, n = 0, \ldots, N\) which are found in the classically allowed mean-field interval

\[
-v < \frac{E_n}{N} < \frac{g}{2} \left( 1 + \frac{v^2}{g^2} \right)
\]

as discussed above. Figure 7 shows the level density

\[
\rho = \frac{\Delta n}{\Delta E}
\]

as a function of the mean-field energy \(E\) for \(N = 1000\) particles and \(v = 1, g = 2\), where the energy interval is discretized in 30 equidistant boxes \(\Delta E\).

Semiclassically the individual quantum energy eigenvalues \(E_n\) can approximately be calculated from the classical action integrals by the Bohr-Sommerfeld quantization scheme (see, e.g., [38]), i.e. by the area \(S(E)\) on the Bloch sphere enclosed by the classical orbit \([7,12]\) and the level density is related to the energy derivative of the action, i.e. the period \(T\), see eq. \((45)\) and \((46)\), which is also shown in the figure (compare also figure \((6)\)). The mean-field period \(T(E)\) diverges for trajectories passing through the saddle point, which agrees with the Viviani case for the chosen parameter values. For energies

![Figure 7](image-url)
above the Viviani energy $E_V = v = 1$ the action area consists of two disconnected loops with the same area and therefore the period is multiplied by a factor of two. The Viviani action $S_V \approx 2.28$ in (12) determines semiclassically the number of states supported by the area of the Viviani window, i.e. the number $N_V \approx S_V(N + 1)/(4\pi) \approx 182$ of states above the Viviani energy $E_V$. Hence we expect $N + 1 - N_V = 819$ states below the Viviani energy.

In the vicinity of this Viviani energy the quantum energy density shows a pronounced maximum [7, 10]. For larger energies we have two almost degenerate states with opposite symmetry. As an example figure 8 shows the Husimi phase space distributions $|\langle \vartheta, \varphi | \Psi_n \rangle|^2$ of the eigenstates $|\Psi_n\rangle$ on the Bloch sphere for $n = 790, 819, 820$ and $837$ with energies $E(n)/N = 0.9733, 1.0007, 1.0017,$ and $1.0158$. The first state localizes on a single closed classical orbit, the second one, $n = 819$, almost exactly at the saddle point in agreement with the semiclassical estimate above. For the other two states the Husimi densities are localized on the two disconnected loops encircling the classical stationary points $s_{1\pm}$ for the corresponding energies.

Let us conclude with a brief discussion of the implications of the classical Viviani curve on the quantum dynamics for the important case when the system is prepared at time $t = 0$ in one of the modes. The left panel of figure 9 shows the nonlinearity dependence of the time evolution of the mean-field population imbalance $z = |\psi_1|^2 - |\psi_2|^2$ (shown in false colors), where the system is prepared in mode 2 corresponding to the
south pole of the Bloch sphere \[39\]. For vanishing nonlinearity one would recover the usual sinusoidal Rabi oscillations between the two modes. For larger nonlinearities the oscillation period increases but one still observes a periodic complete population transfer between the modes. Although the self trapping bifurcation happens at \( g = v \) one only observes a characteristic change in the dynamics at the Viviani critical value of \( g = 2v = 2 \). This can be understood in the following way: At the self trapping transition a saddle point appears, but the orbit passing through the north pole does not reach this point and consequently does not change its characteristics of a complete population transfer. With increasing nonlinearity, however, it coincides with the separatrix orbit at the critical point \( g = 2v \), the period of which diverges, as can be nicely observed in figure 9. For larger values we still find oscillatory behavior, however, \( z \) stays confined to negative values and the system is therefore mainly populating the second mode and performs self-trapping oscillations.

Also shown in figure 9 (right panel) is the corresponding quantum many-particle expectation value \( 2\langle \hat{L}_z \rangle/N \) for \( N = 1000 \) particles and an initial coherent state localized at the south pole. For vanishing interaction the mean-field description is exact but for nonvanishing interactions one observes a decay of the population imbalance, in particular in the vicinity of the critical interaction. The mean-field approximation is still restricted to the Bloch sphere whereas the many-particle angular momentum expectation value can penetrate the sphere. This breakdown of the mean-field approximation \[4, 5\] is a consequence of a mean-field representation by a single phase space point and can be partly cured by the Liouville dynamics approach \[8, 40–42\], i.e. by averaging over an ensemble of initial conditions mimicking the Husimi distribution in classical phase space.
4. Summary

An interesting and unexpected interconnection between contemporary cold atom quantum physics and much older studies in mathematics and astronomy is provided by the celebrated Bose-Hubbard dimer. We have identified the mean-field trajectories of the symmetric Bose-Hubbard dimer as (generalized) Viviani curves or euclidic spherical ellipses. Furthermore we have shown that the dynamics reduces to the oscillation of a mathematical pendulum on a circle with an energy dependent radius as illustrated in figure [1].

In the librational case (ii) the pendulum motion is restricted to an angular region inside the sphere in the $(x,y)$-plane (the full line in the figure). The corresponding motion in the $z$-direction extends from the northern to the southern hemisphere and the spherical ellipse is a single closed loop.

For the rotational pendulum motion (case (i)) the spherical ellipse consists of two loops on the northern and southern hemisphere. The projection on the $z$-axis is restricted to an interval on the positive or negative half-axis, respectively. In the language of the Bose-Hubbard system, $z$ is the population imbalance and therefore the population is trapped in a certain interval, an effect known as self-trapping. The self-trapping transition occurs for $g = v$, i.e. when the radius $r$ of the sphere equals the shift $a$ of the center of the cylinder.

The areas enclosed by the Viviani curves are thus the action integrals needed for a semiclassical quantization of the many-particle spectrum, and govern the energy density. The Viviani curve appears as dynamical separatrix between full oscillations and self-trapping oscillations when the system is prepared in one of the two modes.

Acknowledgments

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References

Bose-Hubbard dimers, Viviani’s windows and pendulum dynamics

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