Driven Quantum Systems (I)

Models for Studying Quantum Chaos

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Introduction

Guided tour through the zoo of chaotic systems

- driven systems with smooth (sinusoidal) driving
- a single degree of freedom (1½ dimensional)
- mixed regular/chaotic phase space
- classical - quantum correspondence
- quantum phase space, entropies, resonance states
Outline

• Introduction
• Basic quantum techniques: A reminder
• The driven anharmonic oscillator
• The driven planar rigid rotor
• Driven Wannier-Stark systems
• Future perspectives: Nonlinear quantum dynamics

see: http:\aleph.physik.uni-kl.de\~korsch
Outline

• Introduction
• Basic quantum techniques: A reminder
  - Space periodic quantum systems: Bloch bands
  - Time periodic quantum systems: Floquet theory
  - Wannier-Stark systems: Bloch oscillations
  - Wannier-Stark ladder of resonances
  - Husimi densities in phase space
• The driven anharmonic oscillator
• The driven planar rigid rotor
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• Future perspectives: Nonlinear quantum dynamics
Basic quantum techniques: (1)

Space periodic quantum systems

\[ H = \frac{p^2}{2m} + V(x), \quad V(x + d) = V(x) \]

\[ H \psi_{n,\kappa}(x) = E_n(\kappa) \psi_{n,\kappa}(x) \]

\[ \psi_{\kappa,n}(x) = e^{i\kappa x} \varphi_{\kappa,n}(x), \quad \varphi_{\kappa,n}(x + d) = \varphi_{\kappa,n}(x) \]

\[ \Rightarrow \psi_{\kappa,n}(x + d) = e^{i\kappa d} \psi_{\kappa,n}(x) \]

**Bloch theorem**

\( \kappa = \) quasimomentum, \( n = \) band index

\( E_n(\kappa) = \) dispersion relation
Basic quantum techniques: (2)

dispersion relation

1\textsuperscript{st} Brillouin zone \quad k_0 = \pi / d
Basic quantum techniques: (3)

Time periodic quantum systems: Floquet theory

\[ \hat{H}(t + T) = \hat{H}(t) \]

symmetries: \[ t \rightarrow t + T \]
\[ t_0 + t \rightarrow t_0 - t \]

Floquet operator \[ \hat{F}(t) = \hat{U}(t + T, t) \]

eigenstates \[ \hat{F}(t)\psi_\alpha(t) = e^{-i\epsilon_\alpha T/\hbar}\psi_\alpha(t) \] quasienergies
Basic quantum techniques: (4)

**Ansatz:** \( \psi_\alpha(t) = e^{-i\epsilon_\alpha t/\hbar} \varphi_\alpha(t) \)

\[ \Rightarrow \varphi_\alpha(t + T) = \varphi_\alpha(t) \]

**Note:** similar to Bloch states for space periodic systems

**Remark:** The \( \varphi_\alpha(t) \) are eigenstates of the quasienergy operator

\[ \hat{K}(t) = \hat{H}(t) - i\hbar \partial_t \]

with eigenvalues \( \epsilon_\alpha \), the quasienergies.
Basic quantum techniques: (5)

The quasienergy spectrum:

- Eigenvalues of the Floquet operator are $e^{-i\epsilon_\alpha \frac{T}{\hbar}}$.
- Quasienergies only defined up to multiples of $\hbar \omega$, $\omega = \frac{2\pi}{T}$.

$\Rightarrow \quad \epsilon_\alpha$ represents a class $\epsilon_{\alpha,n} = \epsilon_\alpha + n\hbar \omega$.

With $-\frac{\hbar \omega}{2} < \epsilon_\alpha \leq \frac{\hbar \omega}{2}$, first Brillouin zone.

Note: For time periodic systems, the quasienergies and quasienergy states take over the rôle of eigenvalues and eigenstates in time independent systems.
Basic quantum techniques: (6)

Wannier-Stark systems (I)

\[ H = \frac{p^2}{2m} + V(x) + Fx \]

\[ V(x + d) = V(d) \]

\( F=0 \):

- space translation operator

\[ T_n \varphi(x) = \varphi(x + nd) \]

- discrete translational symmetry:

\[ [H, \hat{T}_n] = 0 \]

- common eigenstates:

\[ \hat{H} \varphi_{\kappa,\alpha}(x) = E_{\alpha}(\kappa) \varphi_{\kappa,\alpha}(x) \]

\[ \hat{T}_n \varphi_{\kappa,\alpha}(x) = e^{i\kappa x_n} \varphi_{\kappa,\alpha}(x), \quad x_n = nd \]
Basic quantum techniques: (7)

- broken translational symmetry: \[ [\hat{H}, \hat{T}_n] = -Fx_n \hat{T}_n \]

- gauge transformation to accelerated frame:
  \[ H = \frac{(p-Ft)^2}{2m} + V(x) \]
  \[ \hat{U}(t) = \exp\left\{ -\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right\} \]
  \[ \Rightarrow \hat{T}_n \hat{U}(t) = e^{-iFx_n t/\hbar} \hat{U}(t) \hat{T}_n \]

- time evolution operator

- operators commute at multiples of the Bloch time \[ T_B = \frac{2\pi \hbar}{dF} \]

- additional symmetry:
  \[ \hat{U}(T_B) \Phi_{\kappa,\alpha}(x) = e^{-iE_\alpha(\kappa)T_B/\hbar} \Phi_{\kappa,\alpha}(x) \]
  \[ \hat{T}_n \Phi_{\kappa,\alpha}(x) = e^{i\kappa x_n} \Phi_{\kappa,\alpha}(x) \]

\[ \hat{U}(T_B) : \text{Floquet-Bloch operator} \]
Basic quantum techniques: (8)

Bloch Oscillations

- periodic oscillation
- no systematic dispersion
- (within a single band approximation)

Example: motion of a broad gaussian wave packet \( (d = 2\pi) \)
**Basic quantum techniques: (9)**

**Bloch oscillations:** After one Bloch period, the wave packet is reconstructed (up to decay to the upper bands)

![Graph showing Bloch oscillations](image)

- Motion in momentum space
Basic quantum techniques: (10)

Bloch oscillation in coordinate and momentum space
Basic quantum techniques: (11)

Quantum decay and resonance states

Open (scattering) quantum systems -
resonance states = eigenstates \( \psi(x, t) = e^{-i\mathcal{E}t/\hbar} \varphi(x) \)
for outgoing boundary conditions

\[ \Rightarrow \mathcal{E}_n = E_n - i\Gamma_n/2 \]

\( \varphi(x) \longrightarrow \sim e^{\pm ikx} \) for \( x \longrightarrow \pm \infty \)
complex, \( \Gamma_n = \text{width of the state} \)

\[ \Rightarrow |\psi(x, t)|^2 \sim e^{-\Gamma nt/\hbar} \]

\( \tau_n = \hbar/\Gamma_n \) lifetime
Basic quantum techniques: (12)

Computation of complex-energy resonances e.g. by complex scaling techniques

Basic quantum techniques: (13)

**Wannier-Stark systems (II)**

\[ H = \frac{p^2}{2m} + V(x) + Fx \quad V(x + d) = V(d) \]

**Wannier-Stark ladder of resonances**

\[ \hat{H} \Psi_{n,\alpha}(x) = \mathcal{E}_{n,\alpha} \Psi_{n,\alpha}(x) = (\mathcal{E}_{0,\alpha} + ndF_0) \Psi_{n,\alpha}(x) \]

Wannier-Stark states

\[ \mathcal{E}_{n,\alpha} = E_{n,\alpha} - i\Gamma_\alpha/2 \]
Computation of Wannier-Stark resonances

\[ H = \frac{p^2}{2m} + V(x) + Fx \quad V(x + d) = V(x) \]

\[ H_0 \]

gauge transformation to 'momentum frame':

\[ |\psi(t)\rangle = e^{-iFxt/\hbar} |\tilde{\psi}(t)\rangle \]

\[ S(t) \]

shift operator in momentum space:

\[ S(t)|k\rangle = |k - Ft/\hbar\rangle \]

\[ \tilde{H}_0(t) = S^\dagger(t) H S(t) = \frac{(p-Ft)^2}{2m} + V(x) \]

\[ U(T, 0) = S(T) \tilde{U}(T, 0) \]
Basic quantum techniques: (15)

- Shift operator $S(t)$ and space translation $T(d)$ operator commute for $t=T=$ Bloch period.

- New Hamiltonian is explicitly time dependent. Computation of time evolution operator by, e.g.,

$$\tilde{U}(T_B,0) \approx \prod_{j=1}^{J} e^{-i\tilde{H}_0((j-1/2)\Delta t)} , \quad \Delta t = T_B/J$$

in the momentum representation.

- Calculate matrix $U(T_B,0) = S(T_B)\tilde{U}(T_B,0)$ (S is simply a shift of matrix elements).

- Truncation to a finite matrix $\Rightarrow$ not unitary $\Rightarrow$ complex eigenvalues $e^{-i\mathcal{E}_\alpha T_B/\hbar}$ resonance energies $\mathcal{E}_\alpha = E_\alpha - i\Gamma_\alpha/2$
Basic quantum techniques: (16)

Experiment: Bose-Einstein condensate in an optical lattice under a constant force

\[ g \]

Quantum mechanics in phase space

**coherent state** = minimum uncertainty wave packet

position representation:

\[ \langle x|p, q \rangle = \frac{4}{\sqrt{\pi \hbar}} \exp \left( -\frac{s}{2\hbar} (x - q)^2 + \frac{i}{\hbar}px - \frac{i}{\hbar}pq \right) \]

\[ \langle \hat{q} \rangle = q, \quad \langle \hat{p} \rangle = p \]

\[ \Delta q = \sqrt{s\hbar/2}, \quad \Delta p = \sqrt{\hbar/2s} \quad \Rightarrow \quad \Delta p\Delta q = \hbar/2 \]

plane wave with momentum \( p \) weighted by a Gaussian centered at \( q = \) Gaussian wave packet

\( s = \) squeezing parameter
Basic quantum techniques: (18)

\[ \langle p, q | \psi \rangle \sim \int e^{-s(x-q)^2/2\hbar} e^{-ipx/\hbar} \psi(x) \, dx \]

gaussian window at q

Analysis of a function \( \psi(x) \) by projection onto a coherent wave packet with momentum \( p \) centered at \( x=q \)
Basic quantum techniques: (19)

\[ \rho_H(p, q) = |\langle p, q | \psi \rangle|^2 \]

(quasi) phase space density: Husimi function

probability density for finding the quantum particle with momentum \( p \) at point \( q \)

Properties:

- nonnegative and normalized
- invertible (up to a constant)
- is basically determined by its zeros (!)
- can be viewed as a smoothed Wigner density
- violates average laws as
  \[ \int \rho_H(p, q) \, dp = |\psi(q)|^2 \]
Example: Harmonic oscillator

\[ H = \frac{p^2}{2} + \frac{q^2}{2} \]

Husimi distributions for eigenstate \( n=3 \)
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• The driven anharmonic oscillator
  - Classical phase space organization and wave packet dynamics
  - Phase space entropy and quantum localization
  - Destruction of coherence
• The driven planar rigid rotor
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The driven anharmonic oscillator (1)

Quartic oscillator model:

\[ H = \frac{1}{2}p^2 + \frac{1}{4}\beta q^4 - f_0 \cos(\omega t) q \]

units: \( \beta = 1; \omega = 1 \)

stroboscopic Poincaré section

section of a flux tube
The driven anharmonic oscillator (2)

Classical dynamics

Time evolution of a gaussian ensemble of classical particles up to 19 time periods
The driven anharmonic oscillator (3)

Time evolution of a gaussian wavepacket

Quantum dynamics

Husimi phase space distributions
Shannon entropy

\[ S = - \sum_{n=1}^{N} p_n \ln p_n \]

- measure of delocalization of a probability distribution
- can be used in quantum mechanics for an analysis of a quantum state \( \psi \): 
  \[ p_n = |\langle n|\psi\rangle|^2 \]
  (depends on the chosen basis)

Phase space entropy (Wehrl entropy)

\[ \rho(p, q) = |\langle p, q|\psi\rangle|^2 \]

\[ S = -\frac{1}{2\pi\hbar} \int \rho(p, q) \ln \rho(p, q) \, dp \, dq \]
The driven anharmonic oscillator (5)

Time dependence of the phase space entropy:

→ no long time limit; fluctuates with a mean value of

$$\overline{S} = 3.22 < S_{\text{class}}$$
The driven anharmonic oscillator (6)

Destruction of quantum coherence

Recurrence probability of a wave packet started in the chaotic region of phase space compared with a classical ensemble (smoothed) (a)

Same as (a), however distorted by ‘noise’ in the field amplitude (b)
The driven anharmonic oscillator (7)

Quasienergy states $|\alpha\rangle$

Computation:
Diagonalization of the Floquet operator in a harmonic oscillator basis.

Ordering and numbering of states:
Increasing values of $\langle \alpha | H_0 | \alpha \rangle$
The driven anharmonic oscillator (8)

expansion coefficients

expansion coefficients

regular states (localizing on the classical island)

Husimi phase space distributions
The driven anharmonic oscillator (9)

Chaotic states

expansion coefficients

Husimi phase space distribution
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  - Statistics of quasienergies and vector components
  - Rotational excitation
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